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ELEMENTS

OF THE

DIFFERENTIAL AND INTEGRAL

CALCULUS.

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PREFACE.

In preparing the first edition of my Differential and Integral Calculus, published in 1851, my object was to exhibit the elementary principles of this branch of science in a form so simple that they might be distinctly apprehended by beginners who had no special aptitude for mathematical studies. That this attempt was in some measure successful is believed to be indicated by the number of copies of the book (25,000) which have been sold up to 1874. It is, however, well known that within the last few years not only has the number of scientific schools in our country greatly increased, but the courses of instruction have been rendered more thorough and complete. Under these circumstances, I have felt that the former edition of my book was too restricted in its plan, and I decided to rewrite it, so as to accommodate it to the present condition of our scientific schools. In doing this, I have labored to develop the fundamental principles in a method which is strictly logical, and at the same time easily apprehended by students of ordinary capacity. I have also aimed to introduce as great a variety of topics as can be advantageously studied in the time generally devoted to it in our best colleges, and have selected those principles which are best adapted to show the utility of this branch of the Mathematics. For this purpose I have ventured somewhat beyond the limits ordinarily assigned to treatises on the Calculus, and have shown the application of these methods to a few practical questions in Me-

chanics. Throughout the entire work I have aimed to illustrate every principle by appropriate examples designed to test the clearness of the student's conceptions. It is hoped that in its present form the book may be found useful, not only to those who study the Calculus simply as a means of mental discipline, but also to those who may have occasion to apply its principles to the solution of various problems in Mechanics and Astronomy; and that those who wish to prosecute the subject more extensively may find this volume a suitable introduction to more complete treatises on the Calculus.

I take pleasure in acknowledging my special obligations to Professor H. A. Newton for the interest which he has manifested in the revision of this book. He has not only read the entire manuscript, but has also read all the proofs, and has made a very large number of suggestions of which I have studiously availed myself. If the present edition should be found superior to those which have preceded it, the difference must be ascribed in no small degree to the assistance which has thus been rendered me.

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SKETCH OF THE HISTORY

OF

THE DIFFERENTIAL AND INTEGRAL CALCULUS.

THE invention of the Differential and Integral Calculus dates no further back than the seventeenth century, although the questions which led to its discovery must have presented themselves to the earliest geometers. When the ancient geometers wished to compare figures bounded by curved lines with figures bounded by straight lines, they were obliged to resort to indirect and prolix modes of demonstration. Euclid proved that circles are to one another as the squares of their diameters, by proving that the square of one diameter can not be to the square of another diameter, as the one circle to a space either less or greater than the other circle. To accomplish this, he employed inscribed and circumscribed polygons, and proved that if the number of sides of the polygons be continually doubled, the two figures must approach nearer and nearer to equality, and consequently nearer to the intermediate circle. This process has been called the *Method of Exhaustions.*

By methods somewhat similar, Archimedes solved problems still more difficult—such as the determination of the ratio of the sphere to its circumscribing cylinder, the quadrature of the parabola, and the properties of spirals.

For eighteen centuries from the time of Archimedes no important progress was made in this department of science. The next great improvement was made by Cavalieri of Italy, who, in 1635, published a new method, called the *Method of Indivisibles.* Cavalieri considered lines as made up of an infinite number of small points; surfaces as composed of an infinite number of parallel lines, the breadths of which are indefinitely small; and solids as made up of an infinite number of parallel planes, or rather of indefinitely thin plates or laminæ. From this view, it follows that the magnitude of all lines, surfaces, and solids must be proportional to the number of elementary particles they contain; and the only difficulty in comparing magnitudes by this method consists in dividing them into these elements and finding the relative numbers into which they can be divided. The accuracy of this method was disputed by Guldinus in 1640; but the correctness of its results could be demonstrated by the method of exhaustions, and it afforded in many cases a ready means of ascertaining the ratio of curvilinear areas, and of solids bounded by curved surfaces.

About the same time, Roberval in France discovered a method of solving problems relating to curvilinear figures, which did not differ materially from that of Cavalieri.

In 1637, Descartes published a simple method of drawing tangents to all algebraic curves; and by applying Algebra to express the relations of variable quantity, he made an entire revolution in the science of Mathematics, and prepared the way for the grand discoveries which soon followed.

In the *Arithmetic of Infinites*, published by Wallis in 1655, was given the first application of algebraic calculation to quadratures, and this was founded on the method of indivisibles. By this method, Wallis demonstrated the fundamental rule for the quadrature of curves, whose ordinate is proportional to any power of the abscissa. This enabled him to square any curve in which the value of one of the co-ordinates can be expressed in terms of the other, without involving either fractional or negative exponents.

About the year 1666, Newton invented the method of *Fluxions*, by which he was able not only to draw tangents to mechanical curves, but also to solve all problems relating to the quadrature or rectification of curves. According to Newton, all quantities are supposed to be generated by continuous motion: a line, by the motion of a point; a surface, by the motion of a line either constant or variable; and a solid, by the motion of a plane which either retains the same magnitude, or varies according to a certain law. All other quantities, in whatever way they may be expressed, are conceived to be generated in a similar manner. The relative rates or velocities with which magnitudes, and the variables on which they depend, are increasing at any instant, are termed their *fluxions;* and the whole quantities generated in consequence of such velocities of increase are termed the *fluents.*

In 1669, Professor Barrow, of Cambridge, announced to the Secretary of the Royal Society, that Newton had put into his hands a manuscript treatise containing general methods of calculating the quadrature and rectification of curves, and also of resolving equations. In this manuscript the method of fluxions was first indicated; and, through Professor Barrow, these discoveries were communicated not only to English mathematicians, but to several mathematicians on the Continent. This treatise was not published till 1711—more than forty years after it was written.

The first work in which Newton communicated any thing to the public on the subject of fluxions, was in the first edition of the "Principia," in 1687. The principle of his new calculus was there pointed out, but nothing appeared to indicate the peculiar notation which is so essential to this calculus. This first became known to the world in 1693, from the second volume of Wallis's works; and it was not till 1704 that Newton himself published a tract explaining the principles of his method, and applying it to quadratures.

About the same period, the celebrated Leibnitz invented the Differential and Integral Calculus, which corresponds to the direct and inverse method of fluxions. In 1673, Leibnitz visited London and communicated to several members of the Royal Society certain researches in the theory of the differences of numbers, and he was informed that this subject had already been treated by Mouton, a geometer of Lyons. He then turned his attention to the subject of infinite series, and in 1674 he announced to Oldenburg that he had discovered the series which gives the value of a circular arc in terms of its tangent, and that he had very general analytic methods. Oldenburg, in reply, informed him that Newton had discovered methods which give the quadrature of curves, both geometrical and mechanical, and which extended to the circle.

In June, 1676, Newton sent a letter to Oldenburg to be transmitted to Leibnitz. This letter contained his celebrated binomial theorem, and a variety of other matters relating to infinite series and quadratures, but nothing directly relating to the theory of fluxions. In a second letter, written in October, 1676, he explains the manner in which he found the binomial theorem, and suggests the character of his method of fluxions, but conceals the method by transposing the letters of the following sentence: "*An equation containing fluents being given, to find their fluxions and conversely.*" This letter is positive evidence that in 1676 Newton was not only in possession of the principle of his calculus, but that he had already adopted the same nomenclature to which he subsequently adhered.

In June, 1677, Leibnitz sent to Oldenburg, to be communicated to Newton, a letter giving an account of a method which applies to every thing that could be done by the fluxions of Newton. This method was his *differential calculus.* This letter not merely explains the principles of his calculus, but employs the same notation which is in use at the present day. In 1684, Leibnitz first published his method in a tract containing only six pages.

According to the views of Leibnitz, all quantities are supposed to be composed of indefinitely small portions admitting of all possible degrees of relative magnitude. Such portions as are indefinitely small compared to any finite magnitude, are termed *infinitesimals of the first order;* such as are indefinitely small compared to these are called *infinitesimals of the second order*, and so on. Areas are accordingly supposed to be made up of parallelograms having one side finite and another indefinitely small; solids are supposed to be made up of parallelopipeds of finite base but indefinitely small altitudes, etc.; and they are compared with each other by finding the sums of such indefinitely small rectangles, parallelopipeds, etc., by methods which the calculus itself suggests.

Some persons objected to this analysis, because it has the appearance of being only an approximation, and of not coming up to the perfect standard of geometrical precision. Newton endeavored to preserve his calculus from the imputation of throwing any thing away merely because

it was small. Instead of the actual increments of the variable quantities, he employed what he called the fluxions of those quantities; meaning, by fluxions, quantities which had to one another the same ratio which the increments had in their ultimate state. He did not, therefore, reject quantities merely because they were so small that he *might* do so without committing any sensible error, but because he *must* reject them in order to commit no error whatsoever. Fluxions were with him nothing else than measures of the velocities with which variable quantities were supposed to be generated.

When the principles of the differential calculus were attacked, Leibnitz compared his method with that of Archimedes, and showed that it was essentially the same; for instead of supposing the differentials to be in fact infinitely small, it is only necessary to conceive that we may always take them so small that the error which results from the quantities omitted in the calculus shall be less than any given quantity. Indeed, the fluxions and fluents of Newton correspond essentially to the differentials and integrals of Leibnitz; so that the two methods differ only in the notation, and in the peculiar modes of viewing the subject, or what is commonly called the *metaphysics* of the calculus.

The facts which have thus been briefly stated have generally been regarded as proving that Newton was the first inventor of that analysis, which he called by the name of fluxions; but that in the communications made by him or his friends to Leibnitz there was nothing that could convey any idea of the nature of that method, or of the principles on which that analysis was founded. Leibnitz is, therefore, regarded as the second discoverer of the new analysis, but to have made the discovery entirely independent of Newton, and to have an equal claim to originality. Leibnitz has also the merit of having first made known his discovery to the world, having published an account of it in 1684.

Leibnitz enjoyed without contradiction the honor of being the inventor of his calculus until 1699, when a mathematician named Duillier, in a short tract on the curve of swiftest descent, asserted that Newton was the first inventor of the new calculus, and that he would leave to others to decide whether Leibnitz had borrowed any thing from the English geometer. To this charge Leibnitz replied, declaring himself, equally with Newton, to have been the inventor, and asserting his own claim to the first publication of the calculus.

In 1704, when Newton's treatise on the quadrature of curves was published, the Leipsic journalists gave an unfavorable account of it, and intimated that Newton had been led to the idea of fluxions by the differentials of Leibnitz. This charge roused the indignation of the British mathematicians, and in 1708 Keill undertook to prove that the communications of Newton to Leibnitz were sufficient to put the latter in possession of the principles of the new analysis, and that he had only substituted the notion of differentials for that of fluxions.

Leibnitz complained of Keill's proceedings to the Royal Society of London, and the society appointed a commission of its members to draw up a full report of all the communications which had passed between Newton and Leibnitz or their friends on subjects connected with the new analysis. In this report the committee avoided expressing any direct opinion upon the question whether Leibnitz had invented the calculus for himself, or had availed himself of the labors of Newton; but the report seemed to indicate that they were of the latter opinion. Some of the friends of Leibnitz endeavored to fasten upon Newton the charge of plagiarism, and declared that there was no reason to believe that the fluxionary calculus was invented before the differential. Now that the excitement occasioned by this controversy has passed away, mathematicians are unanimous in conceding to Newton the claim of first discoverer; and they also concede that Leibnitz was an independent discoverer. They also concede to Leibnitz the credit of priority in the publication of his new method, and of having invented a superior system of notation, which has entirely superseded the notation introduced by Newton.

In order to attract the attention of mathematicians to the importance of the new analysis, Leibnitz, in 1687, proposed the following problem: "To determine the curve a heavy body must describe in order to descend equally in equal times." This problem was resolved by Huygens and James Bernoulli. In 1690, Bernoulli proposed the problem of the curve which a chain of uniform weight makes when suspended from two points. This problem was resolved by Huygens, Leibnitz, and the proposer. In 1697, John Bernoulli proposed the problem concerning the line along which a body must descend in order to go from one point to another, not perpendicularly under it, in the least possible time. This problem was resolved by Leibnitz, Newton, the two Bernoullis, and De l'Hopital. In 1716, Leibnitz proposed the problem, "A system of curves described according to a known law being given, to describe a curve which shall cut them all at right angles." This problem was resolved by Newton and John Bernoulli. In 1718, Keill, an English mathematician, challenged John Bernoulli to find the path of a projectile moving in a medium in which the resistance varies as the square of the velocity. Bernoulli resolved the problem not only in that particular case, but also when the resistance varies as any power of the velocity. He then required that Keill should produce his own solution; but Keill found his problem too difficult, and maintained a profound silence.

The first elementary treatise in which the principles of the new calculus were explained was the "Analyse des Infiniment Petits," by the Marquis De l'Hopital, published in 1696, from which time a knowledge of the new analysis became generally diffused. It was not, however, every where accepted without opposition. Some mathematicians of considerable eminence in Germany, France, and England objected to the notion of quantities infinitely small. Among those who attacked the new analysis was

Bishop Berkeley. He endeavored to prove that this analysis was inaccurate in its principles, and that if it ever leads to true conclusions, it is from an accidental compensation of errors; and he attempted to fortify his objections by ridicule, calling ultimate ratios the *ghosts of departed quantities*. The objections of Berkeley were fully answered by Jurin and others.

The succession of great geometers was not interrupted by the death of Newton, Leibnitz, and the Bernoullis. Euler and D'Alembert, Lagrange and Laplace, Gauss and Abel, Cauchy and Jacobi have successively directed the onward march of the mathematics. In more than fifty years of incessant thought, Euler wrote 30 separate works, and more than 700 memoirs, which could not altogether be contained in forty large quarto volumes. These writings embrace every existing branch of mathematics, and almost every conceivable application of them; and there is no other mathematician who can be placed near to him in the variety of the subjects which he treated. In all his writings the modern analysis is employed as the instrument of investigation. His "Institutiones Calculi Differentialis" was published in 1755, and his "Institutiones Calculi Integralis," in 3 vols., was published 1768–70.

The mathematical labors of D'Alembert were chiefly directed to questions of Astronomy and Mechanics, but in all he furnished new methods of analysis which contributed greatly to the improvement of the calculus. Among these improvements should be mentioned the method of *partial differences*, invented by him when he was studying the figures which a musical string assumes during its vibrations. He showed that the method of limits furnishes a satisfactory explanation of the theory of fluxions, without admitting the idea of motion, as was done by Newton. He also resolved the problem of the precession of the equinoxes and nutation of the earth's axis, and determined the quantity of each.

The most important of the works of Lagrange is his "Mécanique Analytique," published in 1788, of which a second and enlarged edition was published in 1811. This great work is entirely founded on the *Calculus of Variations*, of which he was the inventor. The whole flows from a single formula, and from a principle known before his time, but whose utility was far from being suspected.

In 1772, Lagrange proposed to show that the theory of the development of functions into series contained the true principles of the differential calculus, independently of the consideration of infinitely small quantities or limits. He called this the calculus of derived functions, and he has developed this view of the subject in his "Théorie des Fonctions Analytiques," published in 1792, and at a later period in his "Leçons sur le Calcul des Fonctions." This system assumes for its basis the expansion of Taylor's theorem, which it establishes by common Algebra.

The great work of Laplace is the "Mécanique Céleste," being a collection of all that had been done by himself or others concerning the theory

of the universe. There is no branch of Physical Astronomy which is not materially indebted to Laplace. Superior to Euler in the power of conquering analytical difficulties, he is almost his equal in the universality of his labors. His "Théorie des Probabilités" exhibits an analysis more refined than any hitherto used; and this work is regarded by some as showing even greater genius than the "Mécanique Céleste."

The great work of Gauss is his "Theoria Motus Corporum Cœlestium," in which are discussed some of the most important problems in Celestial Mechanics, and in which the calculus is only employed as an instrument for the solution of problems which otherwise could not be solved. Gauss also gives an exposition of the method of least squares, which he invented prior to Legendre.

Although Abel died at the age of 27, his brilliant discoveries had already raised the highest expectations of the fruits of his maturer years. He demonstrated the impossibility of the general solution of equations of a higher degree than the fourth; and his papers on Elliptic Functions have been honored with the highest praise.

There is scarcely a branch of the mathematics, either pure or applied, that does not owe something to the labors of Cauchy; but his fame rests chiefly on his residual and imaginary calculus. His published memoirs amount to 500, besides numerous reports on memoirs presented by others. These memoirs treat of the highest subjects in the mathematics—the perfecting and extension of pure analysis. Among his numerous works are his "Course of Analysis," published in 1821; "Applications of the Infinitesimal Calculus to Geometry," published in 1826–28; and his "Differential Calculus," published in 1829.

The chief investigations of Jacobi were connected with the theory of elliptic functions, and his principal work is the "Fundamenta Nova Theoriæ Functionum Ellipticarum," published in 1829; besides which he wrote many special memoirs.

From the preceding sketch, it may be easily understood that the Differential and Integral Calculus, which was first exhibited by Leibnitz in a tract of only six pages, has in less than two centuries grown into a system of theorems and methods sufficient to fill a work of many quarto volumes. The present treatise aims only to develop the first principles of this science. There are numerous works in which a full exposition of the principles and methods of the calculus may be found, among which may be specially mentioned, Bertrand's "Traité de Calcul Différentiel et de Calcul Intégral," 2 vols., 4to.

DIFFERENTIAL CALCULUS.

CHAPTER I.

DEFINITIONS. LIMITING RATIOS. DIFFERENTIATION OF ALGEBRAIC FUNCTIONS.

1. *Classification of quantities.* In the Differential Calculus the quantities employed are divisible into two classes, viz., *constants* and *variables.*

Constant or invariable quantities are such as retain the same values throughout the same discussion, and are usually denoted by the first letters of the alphabet, a, b, c, etc., or by the initial letter of the name of the quantity. Thus in the equation to the circle $y^2=2rx-x^2$, the quantity r, which represents the radius of the circle, undergoes no change in any one operation in which this equation is concerned.

2. *Variable* quantities are such as may assume in the same discussion any value within certain limits determined by the nature of the problem, and are generally represented by the final letters of the alphabet, x, y, z, etc.

Thus in the equation $y^2=2rx-x^2$, the quantities x and y admit of different magnitudes within certain limits: x may have any value between 0 and $2r$; and y may have any value between $+r$ and $-r$.

3. *Functions of one or more variables.* A mathematical expression of any form whatever involving constant and variable quantities is called a *function* of those variable quantities.

Thus the quantities ax^2+bx+c, $\sqrt{2rx-x^2}$, $(a+bx)^n$, etc., are all functions of x.

4. An *explicit* function is one in which the value of the function is directly expressed in terms of the variable and constants. Thus in the equation $y=\sqrt{2rx-x^2}$, y is an explicit function of x.

5. An *implicit* function is one in which the value of the function is not directly expressed in terms of the variable and constants. Thus in the equation $x^3-3xy+y^2=16$, y is an implicit function of x, and x is an implicit function of y. Whenever the equation can be solved with respect to one or both of the variables involved, an implicit function may be made an explicit one.

6. *Notation.* When we wish to denote that y is an explicit function of x, without giving the precise *form* of the function, we employ the notation

$$y=f(x),\ y=\mathrm{F}(x),\ y=\phi(x),\ \text{etc.},$$

or

$$x=f(y),\ x=\mathrm{F}(y),\ x=\phi(y),\ \text{etc.};$$

which expressions are read y is a function of x, or x is a function of y.

When we wish simply to denote that x and y are functions of each other, or that y is an implicit function of x, or x an implicit function of y, we employ the notation

$$f(x,y)=0,\ \mathrm{F}(x,y)=0,\ \text{or}\ \phi(x,y)=0,\ \text{etc.},$$

which is read, function of x and $y=0$. This form represents any equation between two variables with all the terms transposed to the first member.

7. Variables are distinguished as *independent* and *dependent*. An independent variable is a quantity to which we may suppose any value to be arbitrarily assigned. A dependent variable is a quantity whose value varies in consequence of the variation of the independent variable or variables. Thus in the equation to the parabola $y^2=4ax$, if we assign arbitrary values

to x, the corresponding value of y may be known from the equation. In this case x is made the independent variable, and y the dependent variable.

This distinction is made simply for convenience, and either variable may be treated as the independent variable.

8. Functions may be divided into two classes—*Algebraic* and *Transcendental*. An Algebraic function is one in which the relation between the function and its variable can be expressed by a finite number of the ordinary operations of Algebra, viz., addition, subtraction, multiplication, division, the raising to a known power, and the extraction of a known root. Thus in the equation $y=ax^4+bx^2$, y is an algebraic function of x.

Transcendental functions are such as can not be exhibited by means of algebraic expressions containing a finite number of terms. Infinite series and continued fractions which contain the variable, and can not be transformed into algebraic expressions of a finite number of terms, are transcendental functions. They are subdivided into exponential, logarithmic, and trigonometrical.

An *exponential* function is one in which the variable occurs as an exponent, as $y=a^x$, $y=z^x$, etc.

A *logarithmic* function is one which involves the logarithm of a variable expression, as $y=\log. x$, $y=\log. (ax+b)$, etc.

A *trigonometrical* function is one which involves trigonometric operations, as $y=\sin. x$, $y=\tang. x$, etc.

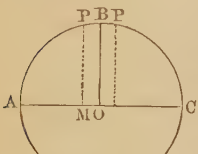
9. Functions are distinguished as *increasing* and *decreasing*. An increasing function is one that increases when the variable increases. Thus in the expressions $y=(a+x)^3$, $y=ax^3$, etc., y is an increasing function of x.

A *decreasing* function is one that decreases when the variable increases. Thus in the expressions $y=\frac{a}{x}$, $y=(a-x)^3$, etc., y is a decreasing function of x.

A function of x may be increasing for certain values of x, and decreasing for other values.

10. *Limiting values of quantities.* The limit of a variable quantity is that value toward which it continually approaches, so that it can be made to differ from that value by less than any assignable quantity.

For example, suppose a regular polygon to be inscribed in a circle. If the number of sides of the polygon be increased, the magnitude of each angle will increase; and if the number of sides be made greater than any finite number, each angle of the polygon will differ from two right angles by less than any assignable quantity. Two right angles is therefore the *limit* toward which each angle of the regular polygon continually approaches when the number of its sides is increased.



Again, let AC be the diameter of a circle, and let any point P be taken on the circumference, from which draw PM perpendicular to AC. If we suppose the point P to move from A toward B, the line PM will continually increase until it becomes equal to BO, the radius of the circle. While the point P moves from B toward C, the line PM will continually decrease until it vanishes at the point C. The radius of the circle is therefore the limit which the perpendicular PM can never exceed.

If we convert $\frac{1}{9}$ into a decimal fraction, it becomes

$$0.1111, \text{etc.},$$

or

$$\tfrac{1}{10} + \tfrac{1}{100} + \tfrac{1}{1000} + \tfrac{1}{10000} + \text{ etc.}$$

Hence the sum of the terms of this series approaches to the value $\frac{1}{9}$, but can never equal it while the number of terms is finite. The limit of the sum of the terms of this series is therefore $\frac{1}{9}$.

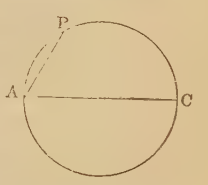
So also the sum of the series

$$1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \tfrac{1}{16} + \text{ etc.},$$

approaches nearer to 2 the greater the number of terms we employ; and by taking a sufficient number of terms, the sum of the series may be made to differ from 2 by less than any quantity we may please to assign. The limit of the sum of the terms of this series is therefore 2.

11. *Value of a limiting ratio.* When two magnitudes decrease simultaneously, they may approach continually toward a ratio of equality or toward some other definite ratio.

Case I. *The two magnitudes may approach a ratio of equality.* Let a point P be supposed to move on the circumference of a circle toward a fixed point A. The arc AP will diminish at the same time with the chord AP, and by bringing the point P sufficiently near to A, we may obtain an arc and its chord, each of which shall be less than any given line, and the arc and its chord continually approach to a ratio of equality. The truth of this proposition will hereafter be proved analytically (Art. 43); at present it may be more satisfactory to the beginner to arrive at the same conclusion by computing the value of this ratio for a variety of arcs. If we begin with an arc of 60°, descending to arcs of 30°, 10°, 5°, 1°, 1′, 1″, etc., and in each case compute both the length of the arc and that of its chord, and thence determine the ratio of the two numbers, we shall obtain the results given below:



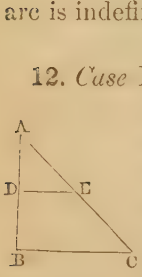
	Chord.	Arc.	Ratio.
60 degrees.....	1.0000000	1.0471976	1.0471976
30 "	0.5176380	0.5235988	1.0115151
10 "	0.1743114	0.1745329	1.0012702
5 "	0.0872388	0.0872665	1.0003178
1 "	0.0174530	0.0174533	1.0000127
1 minute	0.0002909	0.0002909	1.0000000

We find that the chord of one minute does not differ from the

corresponding arc in the first seven decimal places, and the chord of one second does not differ from the arc of one second in the first fifteen decimal places. The arc and its chord are therefore continually approaching to a ratio of equality. The same is true of an arc and its sine, or an arc and its tangent.

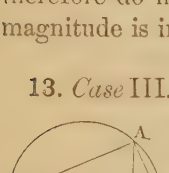
The ratio to which an arc and its chord approach when the arc is indefinitely diminished is called their *limiting ratio.*

12. *Case* II. *Two indefinitely small magnitudes may have some finite ratio different from unity.* Let ABC be a right-angled triangle, having AB equal to BC. Then we have $AB : AC :: 1 : \sqrt{2}$. Let a line be supposed to move from BC toward A, always remaining perpendicular to AB, we shall have the proportion $AD : AE :: 1 : \sqrt{2}$; and however near the line DE approaches to the point A, the ratio of AD to AE remains the same, even though each side of the triangle ADE be made less than any assigned magnitude. In like manner we find that the limiting ratio of a chord to half the corresponding arc is the ratio of 2 to 1. Two magnitudes therefore do not have a ratio of equality simply because each magnitude is indefinitely small.



13. *Case* III. *The ratio of the two magnitudes may increase or decrease without limit.* Let ABC be a triangle inscribed in a semicircle, and let AD be drawn perpendicular to BC. Then (Geom., Bk. IV., Pr. 23) we have

$$BC : AC :: AC : DC.$$



Suppose the point A approaches C, until the chord AC becomes indefinitely small; DC will also become indefinitely small. But the above proportion still continues, and when the ratio of BC to AC becomes indefinitely great, the ratio of AC to DC will also be indefinitely great; that is, although each of two quantities is indefinitely small, their ratio is indefinitely great. Hence we see that when two variable quantities are

supposed to diminish together indefinitely, *their limiting ratio may have any value whatever.*

14. *Increments and their limiting ratio.* When, in an algebraic expression, the variable is supposed to increase, or takes an *increment*, the value of the expression undergoes a change, and the quantity by which its value is changed is called its increment.

Thus, in the expression ax, if x receive the increment h, then ax will become $a(x+h)$, and therefore the increment of ax is $a(x+h)-ax=ah$; that is, if x be increased by h, the function ax will be increased by a times h.

The increment of the independent variable is usually represented by h, and in the following exercises, for the increment of x we shall write incr. x, and for the increment of y shall write incr. y.

Ex. 1. If x receive the increment h, what will be the increment of $y=ax^2$?

Here when x becomes $x+h$, the function ax^2 becomes

$$a(x+h)^2.$$

Hence $$\text{incr. } y=a(x+h)^2-ax^2=2axh+ah^2.$$

Ex. 2. What will be the limiting ratio of incr. y to incr. x in the last example, when h or incr. x approaches zero?

Dividing each side of the last equation by h, we have

$$\frac{\text{incr. } y}{h} \text{ or } \frac{\text{incr. } y}{\text{incr. } x}=2ax+ah.$$

Now this equation must hold true whatever value be given to h. If h be taken very small, ah will also be very small; and the smaller h becomes, the nearer will $2ax+ah$ approach $2ax$. $2ax$ is therefore the limit toward which the value of this ratio approaches as h diminishes.

Hence the limiting value of $\dfrac{\text{incr. } y}{\text{incr. } x}=2ax.$

Ex. 3. If x receive the increment h, required the increment of $y=ax^2+3x$.

When x becomes $x+h$, the function y becomes

$$a(x+h)^2+3(x+h).$$

Hence

$$\text{incr. } y = a(x+h)^2 + 3(x+h) - ax^2 - 3x = 2axh + ah^2 + 3h.$$

Ex. 4. Required the limiting ratio of incr. y to incr. x in the last example.

Dividing each side of the last equation by h, we have

$$\frac{\text{incr. } y}{h} \text{ or } \frac{\text{incr. } y}{\text{incr. } x} = 2ax + ah + 3.$$

When h approaches zero, the term ah becomes indefinitely small, and we have

$$\text{the limiting value of } \frac{\text{incr. } y}{\text{incr. } x} = 2ax + 3.$$

15. *How the value of a limiting ratio is found.* It will be seen from these examples that, in order to discover the limiting ratio of a variable to its function, we ascribe a small increment to the variable, and learn the corresponding increment of the function. We then observe toward what limit the ratio of these increments approaches when the increment of the variable is diminished, which limit is only attained when the increment of the variable is supposed to be zero.

It must not be supposed that the limiting value of a ratio is a mere *approximation;* for, while we speak of h approaching zero, we do this merely to aid our conception of the terms of the ratio; but in finding the value of the limit of the ratio, we actually take h equal to zero. The limiting value of the ratio thus found is therefore not approximately but absolutely exact.

Ex. 5. If x receive the increment h, required the increment of $y = ax^3 - 2bx + c$.

$$\begin{aligned}\text{Incr. } y &= a(x+h)^3 - 2b(x+h) + c - ax^3 + 2bx - c. \\ &= ax^3 + 3ax^2h + 3axh^2 + ah^3 - 2bx - 2bh + c - ax^3 + 2bx - c. \\ &= 3ax^2h + 3axh^2 + ah^3 - 2bh.\end{aligned}$$

Ex. 6. Required the limiting ratio of the increments in the last example.

Dividing each side of the last equation by h, we have

$$\frac{\text{incr. } y}{h} \text{ or } \frac{\text{incr. } y}{\text{incr. } x} = 3ax^2 + 3axh + ah^2 - 2b.$$

Hence the limiting value of $\frac{\text{incr. } y}{\text{incr. } x} = 3ax^2 - 2b$.

Ex. 7. If the side of a square be represented by x, and it be increased by h, by what quantity will the surface of the square be increased? *Ans.* $2xh + h^2$.

Ex. 8. In Ex. 7, what will be the increment given to the diagonal? *Ans.* $h\sqrt{2}$.

Ex. 9. If the radius of a circle be x, and it be increased by h, what will be the increment of the circle? *Ans.* $(2xh + h^2)\pi$.

Ex. 10. Required the limiting ratio of the increments in examples 7, 8, and 9. *Ans.* $2x$; $\sqrt{2}$; and $2\pi x$.

Ex. 11. If x receive the increment h, required the increments of the following functions:

$$y = 3x^4 + b \text{ and } y = (x+1)^2.$$

Ans. $12x^3h + 18x^2h^2 + 12xh^3 + 3h^4$; and $2xh + h^2 + 2h$.

Ex. 12. Required the limiting ratio of the increment of y to the increment of h in the last example.

Ans. $12x^3$; and $2x + 2$.

16. *Notation adopted.* It is important to have some convenient notation for expressing the limiting ratio of two simultaneous increments. In Ex. 4, Art. 14, it was shown that the limiting value of $\frac{\text{incr. } y}{\text{incr. } x} = 2ax + 3$, where $y = ax^2 + 3x$. Now if dx be put for the incr. x in the limiting value, then, in order to maintain a uniform notation, we must put dy for incr. y. We shall then have

$$\frac{dy}{dx} = 2ax + 3;$$

or, substituting the value of y,

$$\frac{d(ax^2 + 3x)}{dx} = 2ax + 3.$$

By $\frac{dy}{dx}$ we therefore understand the limiting value of $\frac{\text{incr. } y}{\text{incr. } x}$; or the limiting value of the ratio of the increment of y to the increment of x when the increment of x approaches zero. The symbol dx is called the *differential* of x, and dy the *dif-*

ferential of y, and the fraction $\frac{dy}{dx}$ is called the *differential coefficient* of y with regard to x. The process by which $\frac{dy}{dx}$ is obtained is called *differentiation*. The letter d placed before any algebraic expression or function indicates therefore that the function is to be differentiated, so that d is a symbol of an *operation* and not of a *quantity*.

17. The *object of the differential calculus* is to determine rules for finding differentials and differential coefficients, and to apply them to the solution of various problems both in the pure and mixed mathematics.

Not only does the symbol $\frac{dy}{dx}$ have an important meaning, but a meaning may also be attached to the symbols dy and dx taken separately. These symbols may be regarded as representing the simultaneous *rates of increase* of a function and of the variable upon which it depends. By the *rate* of increase of a variable quantity, we do not understand its *absolute* increase, but rather what the increase would have been if it had been uniform. This distinction may be illustrated by the example of a falling body. Suppose a stone to fall to the ground from the top of a tower, and that it experiences no resistance from the air. At the commencement of its motion its velocity is zero, but this velocity increases uninterruptedly from second to second, and by the velocity at any instant we understand the space which would have been described in one second, *if* the velocity at that instant could be uniformly maintained for one second.

So also if $y=ax^2+3x$, and we suppose x to increase uniformly, the value of y will not increase uniformly, but at each instant it will have a certain *rate* of increase depending upon the value of x. dy and dx are to be regarded as the simultaneous *rates* of increase of y and x corresponding to any particular value of x. The absolute values of dy and dx are there-

fore indeterminate, and they may be either finite or indefinitely small, provided their ratio is equal to the limiting value of the simultaneous increments of y and x.

18. *General method of differentiation.* From the preceding examples and illustrations we perceive that, in order to find the differential coefficient of any function of a single variable, *we give to the variable any arbitrary increment h, and find the corresponding value of the function, from which we subtract its primitive value. We then divide the remainder by the increment h, and obtain the limit of this ratio by making the increment equal to zero; the result is the differential coefficient. If the differential coefficient be multiplied by the differential of the variable, it will give the differential of the function.*

Ex. 1. If x increase uniformly at the rate of 2 inches per second, at what rate does the value of the expression $2x^2$ increase when x equals 6 inches? *Ans.* 48 inches per second.

Ex. 2. If x increase uniformly at the rate of 3 inches per second, at what rate does the value of the expression $4x^3$ increase when x equals 10 inches? *Ans.*

Ex. 3. If x increase uniformly at the rate of 5 inches per second, at what rate does the value of the expression $2x^4$ increase when x equals 4 inches? *Ans.*

Ex. 4. A ship sailing on a northwest course is found to be making north latitude at the rate of 10 miles an hour. What is her rate of progress? *Ans.* 14.1 miles per hour.

19. *The differential of the product of a variable quantity by a constant is equal to the constant multiplied by the differential of the variable.*

Suppose we have the function

$$y = ax^4.$$

$$\text{Incr. } y = a(x+h)^4 - ax^4 = 4ax^3h + 6ax^2h^2 + 4axh^3 + ah^4;$$

$$\therefore \frac{\text{incr. } y}{h} = 4ax^3 + 6ax^2h + 4axh^2 + ah^3;$$

$$\therefore \text{limiting value of } \frac{\text{incr. } y}{\text{incr. } x} = 4ax^3;$$

that is,
$$\frac{dy}{dx} = 4ax^3,$$

or
$$dy = 4ax^3 dx;$$

that is, the differential of ax^4 is equal to the differential of x^4 multiplied by a. The same process and conclusion will apply if y is a times any function of x.

20. *The differential of a constant term is zero; hence a constant quantity connected with a function of x by the sign plus or minus disappears after differentiation.*

Suppose we have the function

$$y = x^4 + b.$$

$$\text{Incr. } y = (x+h)^4 + b - (x^4 + b);$$

$$= 4x^3h + 6x^2h^2 + 4xh^3 + h^4;$$

$$\therefore \frac{\text{incr. } y}{h} = 4x^3 + 6x^2h + 4xh^2 + h^3;$$

$$\therefore \text{limiting value of } \frac{\text{incr. } y}{\text{incr. } x} = 4x^3;$$

that is,
$$\frac{dy}{dx} = 4x^3,$$

where the constant term b has disappeared in the process of differentiation.

21. *To obtain the differential coefficient of any power of a variable, we must diminish the exponent of the power by unity, and then multiply by the primitive exponent.*

To prove this proposition let us take the function

$$y = x^n.$$

$$\text{Incr. } y = (x+h)^n - x^n, \text{ which, by the binomial theorem,}$$

$$= nx^{n-1}h + \frac{n(n-1)}{2}x^{n-2}h^2 + \text{ etc.};$$

$$\therefore \frac{\text{incr. } y}{h} = nx^{n-1} + \frac{n(n-1)}{2}x^{n-2}h + \text{ etc.};$$

$$\therefore \text{limiting value of } \frac{\text{incr. } y}{\text{incr. } x} = nx^{n-1};$$

that is,
$$\frac{dy}{dx} = nx^{n-1}.$$

Ex. 1. What is the differential of $5ax^3$? *Ans.* $15ax^2dx$.

Ex. 2. What is the differential of $\frac{3}{4}x^2+b$? *Ans.*

Ex. 3. What is the differential of $3x^5$? *Ans.*

Ex. 4. What is the differential of $7a^2x^6+b^3$?

Ex. 5. What is the differential of $4ab^2x^8-c$?

Ex. 6. What is the differential of $3a^3cx^9-d$?

Ex. 7. If the side of an equilateral triangle increase uniformly at the rate of $\frac{1}{2}$ of an inch per second, at what rate is its perpendicular increasing when its side is equal to 8 inches?

Ans. $\frac{\sqrt{3}}{4}$ inch per second.

Ex. 8. If the side of an equilateral triangle increase uniformly at the rate of $\frac{1}{2}$ of an inch per second, at what rate is the area increasing when the side becomes 8 inches?

Ans. $2\sqrt{3}$ inches per second.

Ex. 9. If a circular plate of metal expand by heat so that its diameter increases uniformly at the rate of $\frac{1}{100}$ of an inch per second, at what rate is its surface increasing when the diameter is exactly 2 inches?

Ans. $\frac{\pi}{100}$ inch per second.

Ex. 10. If a circular plate expand so that its area increases uniformly at the rate of $\frac{1}{50}$ of a square inch per second, at what rate is its diameter increasing when the area of the circle is exactly a square inch?

Ans. $\frac{1}{50\sqrt{\pi}}$ inch per second.

Ex. 11. If the diameter of a spherical soap bubble increase uniformly at the rate of $\frac{1}{10}$ of an inch per second, at what rate is its capacity increasing at the moment the diameter becomes 2 inches?

Ans. $\frac{\pi}{5}$ inch per second.

Ex. 12. If the capacity of a spherical soap bubble increase

uniformly at the rate of 2 cubic inches per second, at what rate is the diameter increasing at the moment it becomes 2 inches?

Ans. $\frac{1}{\pi}$ inch per second.

Ex. 13. If the diameter of a circular plate expand uniformly at the rate of $\frac{1}{10}$ of an inch per second, what is the diameter of the circle when its area is expanding at the rate of a square inch per second?

Ans. $\frac{20}{\pi}$ inches.

Ex. 14. If the diameter of a sphere increase uniformly at the rate of $\frac{1}{10}$ of an inch per second, what is its diameter when the capacity is increasing at the rate of 5 cubic inches per second?

Ans. $\frac{10}{\sqrt{\pi}}$ inches.

Ex. 15. If the diameter of the base of a cone increase uniformly at the rate of $\frac{1}{10}$ inch per second, at what rate is its solidity increasing when the diameter of the base becomes 10 inches, the height being constantly 1 foot?

Ans. 2π inches per second.

Ex. 16. In a parabola whose equation is $y^2=4x$, if x increase uniformly at the rate of 1 inch per second, at what rate does y increase when the abscissa equals 9 inches?

Ans. $\frac{1}{3}$ inch per second.

Ex. 17. In the same parabola, at what point do the abscissa and ordinate vary at the same rate? *Ans.* When $y=2$.

Ex. 18. In an ellipse whose equation is $9x^2+16y^2=144$, at what point of the curve does the abscissa decrease at the same rate that the ordinate increases?

Ans. When $x=3.2$, and $y=1.8$.

22. *The differential of the sum or difference of any number of functions dependent on the same variable is equal to the sum or difference of their differentials taken separately.*

Let y and z denote two functions of x, and u their sum. Suppose that y', z', u' denote the values these functions assume when x is changed into $x+h$. Then

$$u=y+z,$$

and $$u'=y'+z'.$$

Therefore $$u'-u=y'-y+z'-z;$$

that is, $$\text{incr. } u=\text{incr. } y+\text{incr. } z.$$

Dividing by h, $$\frac{\text{incr. } u}{h}=\frac{\text{incr. } y}{h}+\frac{\text{incr. } z}{h}.$$

This equation being true for all values of h, it will also be true when h approaches zero; therefore, taking the limiting values by putting the ratios of the differentials for the ratios of the increments, we have

$$\frac{du}{dx}=\frac{dy}{dx}+\frac{dz}{dx},$$

or $$du=dy+dz.$$

So, also, if $$u=y-z,$$

we shall have $$du=dy-dz.$$

Ex. 1. What is the differential of $6x^4-5x^3-2x$?

Ans. $(24x^3-15x^2-2)dx.$

Ex. 2. What is the differential of ax^2-cx?

Ans. $(2ax-c)dx.$

Ex. 3. What is the differential of $3ax^3-bx^4$?

Ans. $(9ax^2-4bx^3)dx.$

Ex. 4. What is the differential of $a^6+3a^4x^2+3a^2x^4+x^6$?

Ans. $(6a^5x+12a^3x^3+6x^5)dx.$

Ex. 5. What is the differential of $5x^3-3x^2+6x+2$?

Ans. $(15x^2-6x+6)dx.$

Ex. 6. What is the differential of $7x^5+6x^3-5ax^2+3x-6$?

Ans. $(35x^4+18x^2-10ax+3)dx.$

Ex. 7. In the function $y=2x^3+x^2$ find the value of x when y is increasing 60 times as fast as x. *Ans.* $x=3$, or $-3\frac{1}{3}$.

23. If u represent any function of x, and we change x into $x+h$, *the new value of the function will consist of three parts:*

1st. *The primitive function u.*

2d. *The differential coefficient of the function multiplied by the first power of the increment h.*

3d. *A function of x and h multiplied by the second power of the increment h.*

We have seen in the preceding examples that when u is a function of x, and we change x into $x+h$, the new value of the function consists of a series of terms which may be arranged in the order of the ascending powers of h, and the development is of the following form:

$$u'=u+Ah+Bh^2+Ch^3+\text{ etc.},$$

or

$$u'=u+Ah+h^2(B+Ch+\text{ etc.}).$$

If we represent $B+Ch+$ etc., by B' where B' is a function both of x and h, we have

$$u'=u+Ah+B'h^2. \qquad (1)$$

Transposing, and dividing by h, we find

$$\frac{u'-u}{h}=A+B'h,$$

or

$$\frac{\text{incr. } u}{\text{incr. } x}=A+B'h;$$

$$\therefore \text{limiting value of } \frac{\text{incr. } u}{\text{incr. } x}=A,$$

or

$$\frac{du}{dx}=A;$$

that is, A is the differential coefficient of the function.

We see from equation (1) that u', the new value of the function, consists of the primitive function u, plus the differential coefficient of the function multiplied by h, plus a function of x and h multiplied by h^2.

This expression for the new value of the function will be referred to hereafter under the form

$$u'=u+Ah+Bh^2. \qquad (2)$$

It is here assumed that the function of $x+h$ can be developed into series in powers of h. If any function u of $x+h$ shall be found that can not be developed into the form

$$u'=u+Ah+Bh^2+Ch^3+\text{ etc.},$$

where the *etc.* includes only higher powers of h, the proposition of this Article will not be regarded as proved for such function.

24. *The differential of the product of two functions dependent on the same variable is equal to the sum of the products obtained by multiplying each function by the differential of the other.*

Let us designate two functions by u and v, and suppose them to depend on the same variable x; then, when x is increased so as to become $x+h$, the new functions, Art. 23, may be written

$$u'=u+\mathrm{A}h+\mathrm{B}h^2,$$
$$v'=v+\mathrm{A}'h+\mathrm{B}'h^2.$$

If we multiply together the corresponding members of these equations, we shall have

$$u'v'=uv+\mathrm{A}vh+\mathrm{B}vh^2$$
$$+\mathrm{A}'uh+\mathrm{A}\mathrm{A}'h^2+\text{ etc.},$$
$$+\mathrm{B}'uh^2+\text{ etc.},$$

where, it will be observed, the terms omitted contain powers of the increment higher than h^2.

Transposing, and dividing by h, we have

$$\frac{u'v'-uv}{h}=\mathrm{A}v+\mathrm{A}'u+\text{other terms containing } h.$$

When we make h equal to zero, Art. 15, the terms containing h disappear, and we have

$$\frac{d(uv)}{dx}=\mathrm{A}v+\mathrm{A}'u,$$

or, multiplying by dx,

$$d(uv)=v\mathrm{A}dx+u\mathrm{A}'dx.$$

But $\mathrm{A}dx$ is equal to du,

and $\mathrm{A}'dx$ is equal to dv;

hence
$$d(uv)=vdu+udv, \qquad (1)$$

which was the proposition to be demonstrated.

25. If we divide both members of equation (1), Art. 24, by uv, we shall have

$$\frac{d(uv)}{uv}=\frac{du}{u}+\frac{dv}{v};$$

that is, *the differential of the product of two functions divided by their product is equal to the sum of the quotients*

obtained by dividing the differential of each function by the function itself.

Ex. 1. What is the differential of xy^2? *Ans.* $y^2dx+2xydy$.

Ex. 2. What is the differential of x^2y^2? *Ans.*

Ex. 3. What is the differential of ax^2y^3? *Ans.*

Ex. 4. What is the differential of $ax^3(x^2+2b)$?

Ans.

Ex. 5. What is the differential of $(x^2+a)\,(2x+b)$?

Ans.

Ex. 6. What is the differential of $(x^3+a)\,(3x^2+b)$?

Ans.

Ex. 7. What is the differential of $(a+2x^2)\,(a-3x^3)+6x^5$?

Ans.

Ex. 8. What is the differential of $(1-x^3)\,(1+x+x^3)+x^5-1$?

Ans.

26. *The differential of the product of any number of functions of the same variable is equal to the sum of the products obtained by multiplying the differential of each function by the product of all the other functions.*

Let us designate three functions by u, v, and z, and suppose them to depend on the same variable x. Substitute y for vz, and we shall have $uvz=uy$,

and $d(uvz)=d(uy)$.

But, by Art. 24, $d(uy)=ydu+udy$; (1)

and since $y=vz$, we have, by the same Art.,

$dy=zdv+vdz$.

Substituting these values of y and dy in equation (1), it becomes $d(uvz)=vzdu+uzdv+uvdz$. (2)

The same method is applicable to the product of four or more functions.

27. If we divide both members of equation (2), Art. 26, by uvz, we shall have $\frac{d(uvz)}{uvz}=\frac{du}{u}+\frac{dv}{v}+\frac{dz}{z}$,

which is an extension of Art 25.

Ex. 1. What is the differential of xy^2z?

Ans. $y^2zdx+2xyzdy+xy^2dz$.

Ex. 2. What is the differential of $x^2y^2z^3$?

Ans. $2xy^2z^3dx+2x^2yz^3dy+3x^2y^2z^2dz$.

Ex. 3. What is the differential of axy^3z^3?

Ans. $ay^3z^3dx+3axyz^3dy+3axy^3z^2dz$.

Ex. 4. What is the differential of $x(x^2+a)(x+2b)$?

Ans. $4x^3dx+6bx^2dx+2a^2xdx+3abx^2dx+2a^2bdx$.

Ex. 5. What is the differential of $ax^2(x^3+a)(x+3b)$?

Ans. $6ax^5dx+15abx^4dx+8a^2x^3dx+12a^2bx^2dx+6a^3xdx+3a^3bdx$.

Ex. 6. What is the differential of $x^2(1+ax^2)(1-ax^2)$?

Ans. $(2x-6a^2x^5)dx$.

28. To find the differential of a fraction, *multiply the denominator by the differential of the numerator; subtract the product of the numerator by the differential of the denominator, and divide the remainder by the square of the denominator.*

Let us designate the fraction by $\frac{u}{v}$, and suppose

$$\frac{u}{v}=y; \tag{1}$$

then $$u=vy.$$

Therefore, by Art. 24, $$du=ydv+vdy;$$

whence $$vdy=du-ydv. \tag{2}$$

Substituting, in the second member of equation (2), the value of y from equation (1), we have

$$vdy=du-\frac{udv}{v}.$$

Dividing by v, we obtain

$$dy=\frac{vdu-udv}{v^2};$$

that is, $$d\left(\frac{u}{v}\right)=\frac{vdu-udv}{v^2},$$

which was the proposition to be demonstrated.

29. If the numerator of the fraction is constant, its differential will be zero (Art. 20), and we shall have

$$d\left(\frac{c}{v}\right)=-\frac{cdv}{v^2};$$

that is, to find the differential of a fraction whose numerator is constant, *multiply the numerator with its sign changed by the differential of the denominator, and divide by the square of the denominator.*

Ex. 1. What is the differential of $\frac{x^2}{y^3}$?

$$Ans.\ \frac{2xy^3dx-3x^2y^2dy}{y^6},\ \text{or}\ \frac{2xydx-3x^2dy}{y^4}.$$

Ex. 2. What is the differential of $\frac{a}{x^n}$? $Ans.$

Ex. 3. What is the differential of $\frac{c}{ax^3}$? $Ans.$

Ex. 4. What is the differential of $\frac{x}{1-x}$? $Ans.$

Ex. 5. What is the differential of $\frac{1+x^2}{1-x^2}$? $Ans.$

Ex. 6. What is the differential of $\frac{a^2+x^2}{b^2-x^2}$? $Ans.$

Ex. 7. What is the differential of $\frac{x^3}{x^2-1}-\frac{x^2}{x-1}$?

$$Ans.\ \frac{2xdx}{(x^2-1)^2}.$$

Ex. 8. What is the differential of $ax^3-\frac{x^3}{x^3-x+c}$?

$$Ans.\ 3ax^2dx-\frac{x^2(3c-2x)dx}{(x^3-x+c)^2}.$$

30. *The differential of a function affected with any exponent whatever is the continued product of the exponent, the function itself with its exponent diminished by unity, and the differential of the function.*

This proposition is the same as Art. 21, and the demonstration there given may be regarded as general, since the bino-

mial theorem is true whether the exponent of the power be positive or negative, integral or fractional. This theorem may, however, be deduced directly from Art. 27.

Let it be required to find the differential of x^n, where the exponent n may be either positive or negative, integral or fractional.

Case I. When n is positive and a whole number.

x^n may be considered as the product of n factors each equal to x. Hence, by Art. 27,

$$\frac{d(x^n)}{x^n}=\frac{d(xxxx\ldots.)}{xxxx\ldots.}=\frac{dx}{x}+\frac{dx}{x}+\frac{dx}{x}+\frac{dx}{x}+\ldots.$$

and since there are n equal factors in the first member of the equation, there will be n equal terms in the second; hence

$$\frac{d(x^n)}{x^n}=\frac{ndx}{x},$$

or

$$d(x^n)=nx^{n-1}dx.$$

Case II. When n is a positive fraction.

Represent the fraction by $\frac{r}{s}$, and let

$$u=x^{\frac{r}{s}}.$$

Raising each member of the equation to the power s, we shall have

$$u^s=x^r,$$

and since r and s are supposed to represent entire numbers, we shall have, by Case I.,

$$su^{s-1}du=rx^{r-1}dx.$$

Now, since $u=x^{\frac{r}{s}}$, we have $u^{s-1}=x^{\frac{rs-r}{s}}$. Substituting this in the last equation, we have

$$sx^{\frac{rs-r}{s}}du=rx^{r-1}dx,$$

whence

$$du=\frac{r}{s}x^{\frac{r-s}{s}}dx=\frac{r}{s}x^{\frac{r}{s}-1}dx,$$

which is of the same form as $nx^{n-1}dx$, substituting $\frac{r}{s}$ for n.

Case III. When n is negative, either integral or fractional.

Suppose

$$u=x^{-n},$$

which may be written $u=\frac{1}{x^n}$.

Differentiating by Art. 29, we have

$$du=\frac{-d(x^n)}{x^{2n}};$$

and differentiating the numerator by Case I., or by Case II. if n represent a fraction, we have

$$du=-\frac{nx^{n-1}dx}{x^{2n}},$$

or, reducing, we have

$$du=-nx^{-n-1}dx,$$

which is of the same form as $nx^{n-1}dx$, by substituting $-n$ for $+n$.

The proposition above enunciated may therefore be considered general, whatever be the exponent of x.

Ex. 1. What is the differential of ax^{n+1}?

Ans. $a(n+1)x^n dx$.

Ex. 2. What is the differential of $\frac{a}{b}x^{\frac{1}{n}}+c$? *Ans.*

Ex. 3. What is the differential of $ab^2x^{\frac{2}{3}}$? *Ans.*

Ex. 4. What is the differential of $bx^{\frac{2}{5}}$? *Ans.*

Ex. 5. What is the differential of cx^{-3}? *Ans.*

Ex. 6. What is the differential of $x^2y^{\frac{1}{2}}z^3$? *Ans.*

Ex. 7. If the area of a square increase uniformly at the rate of $\frac{1}{10}$ of a square inch per second, at what rate is the side increasing when the area is 100 square inches? *Ans.*

Ex. 8. If the solidity of a cube increase uniformly at the rate of a cubic inch per second, at what rate is the edge increasing when the solid becomes a cubic foot? *Ans.*

31. *The differential of the square root of a variable quantity is equal to the differential of that quantity divided by twice the radical.*

Let it be required to find the differential of

$$\sqrt{x} \text{ or } x^{\frac{1}{2}}.$$

According to the preceding article,

$$d(x^{\frac{1}{2}}) = \frac{1}{2}x^{\frac{1}{2}-1}dx,$$

$$= \frac{1}{2}x^{-\frac{1}{2}}dx,$$

$$= \frac{dx}{2\sqrt{x}}.$$

The preceding proposition is only a particular case of Art. 30; but we have so frequent occasion to differentiate radicals of the second degree, that a special rule for this case is desirable.

Ex. 1. What is the differential of $\sqrt{ax^3}$?

Ans. $\frac{3ax^2dx}{2\sqrt{ax^3}}$, or $\frac{3}{2}a^{\frac{1}{2}}x^{\frac{1}{2}}dx$.

Ex. 2. What is the differential of $\sqrt{abx^7}$? *Ans.*

Ex. 3. What is the differential of $\sqrt{ax^5}$? *Ans.*

Ex. 4. What is the differential of $a\sqrt{x} - \frac{x}{3}$?

Ans.

Ex. 5. What is the differential of $\sqrt{ax} + \sqrt{c^2x^3}$?

Ans.

Ex. 6. What is the differential of $\frac{x^2}{\sqrt{1+x^4}}$?

Ans. $\frac{2xdx}{(1+x^4)^{\frac{3}{2}}}$.

32. *Polynomial raised to any power.* Art. 30 extends to any function of x affected with any exponent. Let us suppose this function to be a polynomial, and let it be required to differentiate the function

$$u = (ax+x^2)^n.$$

Substitute y for $ax+x^2$, and we have

$$u = y^n.$$

Whence, by Art. 30, $du = ny^{n-1}dy$.

Restoring the value of y, we have

$$du = n(ax+x^2)^{n-1}d(ax+x^2).$$

From which we see that to obtain the differential of a polynomial raised to any power, *we must diminish the exponent of the power by unity, and then multiply by the primitive exponent, and by the differential of the polynomial.*

In the preceding example the differentiation of $ax+x^2$ is only indicated. If we actually perform the differentiation, we shall have

$$du=n(ax+x^2)^{n-1}(a+2x)dx.$$

Ex. 1. What is the differential of $\sqrt{a+bx^2}$?

Ans. $\dfrac{bxdx}{\sqrt{a+bx^2}}$.

Ex. 2. What is the differential of $(ax^2+x^3)^3$?

Ans.

Ex. 3. What is the differential of $\sqrt{ax+bx^2+cx^3}$?

Ans.

Ex. 4. What is the differential of $(ax-x^2)^5$?

Ans.

Ex. 5. What is the differential of $(a+bx^2)^{\frac{5}{4}}$?

Ans.

Ex. 6. What is the differential of $(a+x^2)^{\frac{1}{n}}$?

Ans.

Ex. 7. If x increase uniformly at the rate of $\frac{1}{100}$ of an inch per second, at what rate is the expression $(1+x)^3$ increasing when x equals 9 inches?

Ans.

Ex. 8. A boy standing on the top of a tower whose height is 60 feet, observed another boy running toward the foot of the tower at the rate of five miles an hour on the horizontal plane; at what rate is he approaching the first when he is 80 feet from the foot of the tower? *Ans.* 4 miles an hour.

Ex. 9. In the last example how far was the boy from the foot of the tower when he was approaching the foot twice as fast as he approached the top? *Ans.* $20\sqrt{3}$.

33. *To differentiate a function of a function.* Let u be a function of y, and y a different function of x; that is,

$$u=f(y) \qquad (1)$$

and $$y=\phi(x); \qquad (2)$$

it is required to find the differential coefficient of u regarded as a function of x.

In the equation $y=\phi(x)$ let x be increased by h; then y will receive an increment $y'-y$, which we will denote by k. When we substitute $x+h$ for x in equation (2), we have (Art. 23)

$$y'=y+Ah+Bh^2+ \text{ etc.};$$

or

$$\frac{y'-y}{h}=A+Bh+ \text{ etc.}, \qquad (3)$$

where A is equal to $\frac{dy}{dx}$.

Also, when we substitute $y+k$ for y in equation (1), we have

$$u'=u+Mk+Nk^2+ \text{ etc.};$$

or

$$\frac{u'-u}{k}=M+Nk+ \text{ etc.}, \qquad (4)$$

where M is equal to $\frac{du}{dy}$.

Multiplying together equations (3) and (4), member by member, we have

$$\frac{u'-u}{k}\times\frac{y'-y}{h}=AM+ANk+BMh+ \text{ etc.}$$

But since, by hypothesis, $y'-y=k$, we have

$$\frac{u'-u}{h}, \text{ or } \frac{\text{incr. } u}{\text{incr. } x}=AM+ANk+BMh+ \text{ etc.}$$

Now when h is made $=0$, k will also $=0$, since k denotes the increment of y caused by ascribing the increment h to x. Hence for the limiting ratio we shall have

$$\frac{du}{dx}=AM.$$

But $A=\frac{dy}{dx}$ and $M=\frac{du}{dy}$; hence

$$\frac{du}{dx}=\frac{du}{dy}\times\frac{dy}{dx};$$

that is, *the differential coefficient of u, regarded as a function*

of x, is equal to the differential coefficient of u regarded as a function of y, multiplied by the differential coefficient of y regarded as a function of x.

Ex. 1. Given $u=3y^2$ and $y=x^3+ax^2$ to find $\frac{du}{dx}$.

$$\frac{du}{dy}=6y \text{ and } \frac{dy}{dx}=3x^2+2ax;$$

hence $$\frac{du}{dx}=6y(3x^2+2ax)=6(x^3+ax^2)(3x^2+2ax).$$

Ex. 2. Differentiate the function $u=\sqrt{a^2-x^2}$.

Assume $y=a^2-x^2$; we then have the two functions

$$u=y^{\frac{1}{2}} \text{ and } y=a^2-x^2;$$

hence $$\frac{du}{dx}=\frac{-x}{\sqrt{a^2-x^2}}.$$

Ex. 3. Differentiate the function $u=(a+bx^m)^n$.

$$Ans.\ \frac{du}{dx}=bmnx^{m-1}(a+bx^m)^{n-1}.$$

Ex. 4. Differentiate the function $u=(a+\sqrt{x})^3$.

$$Ans.\ \frac{du}{dx}=\frac{3(a+\sqrt{x})^2}{2\sqrt{x}}.$$

34. The preceding principles will enable us to differentiate all algebraic functions of one independent variable, whether explicit or implicit, however complicated they may be, because all the combinations of quantities that can be formed by the common operations of Algebra have been considered in the demonstrations.

Ex. 1. What is the differential of the function $u=\frac{a+x}{a+x^2}$?

According to Art. 28,

$$du=\frac{(a+x^2)dx-2x(a+x)dx}{(a+x^2)^2}$$

which may be reduced to

$$du=\frac{(a-2ax-x^2)dx}{(a+x^2)^2}.$$

Ex. 2. Differentiate the function $u=\sqrt{x^2+a\sqrt{x}}$.

Ans.

Ex. 3. Differentiate the function $u=\dfrac{(b+x)^2}{x}$.

Ans.

Ex. 4. Differentiate the function $u=\dfrac{x^2}{(a+x^3)^2}$.

Ans.

Ex. 5. Differentiate the function $u=\dfrac{1}{(a+x^2)^n}$.

Ans.

Ex. 6. What is the differential of the function

$$u=x(a+x)(a^2+x^2)\ ?$$

Ans. $du=(a^3+2a^2x+3ax^2+4x^3)dx.$

Ex. 7. What is the differential of the function

$$u=(a+bx)^2(m+nx)^3\ ?$$

Ans. $du=3n(a+bx)^2(m+nx)^2dx+2b(a+bx)(m+nx)^3dx.$

Ex. 8. What is the differential of the function

$$u=(a+bx^2)^3(c+ex^4)^5\ ?$$

Ans. $du=20(a+bx^2)^3(c+ex^4)^4ex^3dx+6b(c+ex^4)^5(a+bx^2)^2xdx.$

Ex. 9. What is the differential of the function

$$u=\sqrt{4x^3-5}\ ?$$

Ans. $du=\dfrac{6x^2dx}{\sqrt{4x^3-5}}.$

Ex. 10. What is the differential of the function

$$u=x(a^2+x^2)\sqrt{a^2-x^2}\ ?$$

Ans. $du=\dfrac{(a^4+a^2x^2-4x^4)dx}{\sqrt{a^2-x^2}}.$

Ex. 11. What is the differential of the function

$$u=\frac{a+x}{\sqrt{a-x}}\ ?$$

Ans. $du=\dfrac{3a-x}{2(a-x)^{\frac{3}{2}}}dx.$

Ex. 12. What is the differential of the function

$$u=\frac{x}{x+\sqrt{1-x^2}}?$$

$$Ans.\ du=\frac{dx}{\sqrt{1-x^2}+2x(1-x^2)}.$$

Ex. 13. What is the differential of the function

$$u=a+\frac{4\sqrt{x}}{3+x^2}?$$

$$Ans.\ du=\frac{6(1-x^2)dx}{(3+x^2)^2\sqrt{x}}.$$

Ex. 14. What is the differential of the function

$$u=\frac{a^2-x^2}{a^4+a^2x^2+x^4}?$$

$$Ans.\ du=\frac{-2x(2a^4+2a^2x^2-x^4)dx}{(a^4+a^2x^2+x^4)^2}.$$

Ex. 15. What is the differential of the function

$$u=\frac{x^n}{(1+x)^n}?$$

$$Ans.\ du=\frac{nx^{n-1}dx}{(1+x)^{n+1}}.$$

Ex. 16. What is the differential of the function

$$u=(a-x)\sqrt{a^2+x^2}?$$

$$Ans.\ du=-\frac{(a^2-ax+2x^2)dx}{\sqrt{a^2+x^2}}.$$

Ex. 17. What is the differential of the function

$$u=(a^2-x^2)\sqrt{a+x}?$$

$$Ans.\ du=\frac{1}{2}(a-5x)(a+x)^{\frac{1}{2}}dx.$$

Ex. 18. What is the differential of the function

$$u=\frac{ay}{\sqrt{x^2+y^2}}?$$

$$Ans.\ du=-\frac{axydx-ax^2dy}{(x^2+y^2)^{\frac{3}{2}}}.$$

Ex. 19. What is the differential of the function

$$u=(2a^2+3x^2)(a^2-x^2)^{\frac{3}{2}}?$$

$$Ans.\ du=-15x^3(a^2-x^2)^{\frac{1}{2}}dx.$$

Ex. 20. What is the differential of the function

$$u=\frac{\sqrt{a+x}+\sqrt{a-x}}{\sqrt{a+x}-\sqrt{a-x}}?$$

$$Ans.\ du=-\frac{a^2+a\sqrt{a^2-x^2}}{x^2\sqrt{a^2-x^2}}dx.$$

Ex. 21. What is the differential of the function

$$u=\frac{a+2bx}{(a+bx)^2}?$$

$$Ans.\ du=\frac{-2b^2xdx}{(a+bx)^3}.$$

CHAPTER II.

DIFFERENTIATION OF TRANSCENDENTAL FUNCTIONS.

35. A transcendental function (Art. 8) is one in which the relation between the function and its variable can not be exhibited by means of algebraic expressions containing a finite number of terms.

Thus a^x, $a \log. x$, sin. x, tang. x, etc., are transcendental functions; the first is called an exponential function, the second a logarithmic function, while the third and fourth are trigonometrical functions.

36. *To find the differential of a logarithm.* Let it be required to differentiate $y = \log. x$.

It is shown in Algebra, Art. 425, that

$$\log. (1+m) = \mathrm{M}(m - \frac{m^2}{2} + \frac{m^3}{3} - \text{etc.}),$$

where M represents the modulus of the system of logarithms, and m represents any quantity whatever. Substituting for m the quantity $\frac{h}{x}$, we obtain

$$\log. \left(\frac{x+h}{x}\right) = \mathrm{M}\left(\frac{h}{x} - \frac{h^2}{2x^2} + \frac{h^3}{3x^3} - \text{etc.}\right).$$

But since the logarithm of a quotient is equal to the logarithm of the dividend diminished by the logarithm of the divisor, we have

$$\log. (x+h) - \log. x = \mathrm{M}\left(\frac{h}{x} - \frac{h^2}{2x^2} + \frac{h^3}{3x^3} - \text{etc.}\right);$$

and dividing by h, we have

$$\frac{\log. (x+h) - \log. x}{h} = \mathrm{M}\left(\frac{1}{x} - \frac{h}{2x^2} + \frac{h^2}{3x^3} - \text{etc.}\right),$$

which is a general expression for the ratio of the increment of the function to that of the variable; that is

$$\frac{\text{incr. } y}{\text{incr. } x}=\mathrm{M}\left(\frac{1}{x}-\frac{h}{2x^2}+\frac{h^2}{3x^3}-\text{ etc.}\right).$$

Hence the limiting value of

$$\frac{\text{incr. } y}{\text{incr. } x}=\frac{\mathrm{M}}{x};$$

that is,

$$\frac{dy}{dx}=\frac{d\log. x}{dx}=\frac{\mathrm{M}}{x},$$

or

$$d\log. x=\frac{\mathrm{M}dx}{x};$$

that is, *the differential of the logarithm of a quantity is equal to the modulus of the system into the differential of the quantity divided by the quantity.*

37. Throughout this treatise, Naperian logarithms will always be used, unless the contrary is stated. Denoting the Naperian logarithm by log., and remembering that the modulus of this system is unity (Algebra, Art. 426), we have

$$d\log. x=\frac{dx}{x};$$

that is, *the differential of the Naperian logarithm of any quantity is equal to the differential of the quantity divided by the quantity.*

Ex. 1. Differentiate the function $y=\log.(ax^3)$.

$$Ans.\ dy=\frac{3dx}{x}.$$

Ex. 2. Differentiate the function $y=\log.(a+x)$.

$$Ans.\ dy=\frac{dx}{a+x}.$$

Ex. 3. Differentiate the function $y=\log.(a+x^2)$.

$$Ans.\ dy=\frac{2xdx}{a+x^2}.$$

Ex. 4. Differentiate the function $y=\log.(ax^n)$.

$$Ans.\ dy=\frac{ndx}{x}.$$

Ex. 5. Differentiate the function $y=\log.\frac{x}{\sqrt{a^2+x^2}}$.

Ans. $dy=\frac{a^2dx}{x(a^2+x^2)}$.

Ex. 6. Differentiate the function $y=\log.\frac{a+x}{a-x}$.

Ans. $dy=\frac{2adx}{a^2-x^2}$.

Ex. 7. Differentiate the function $y=\log.\left\{x+(x^2-1)^{\frac{1}{2}}\right\}$.

Ans. $dy=\frac{dx}{\sqrt{x^2-1}}$.

Ex. 8. Differentiate the function $y=\log.\frac{\sqrt{x^2+1}-1}{\sqrt{x^2+1}+1}$.

Ans. $dy=\frac{2dx}{x\sqrt{x^2+1}}$.

Ex. 9. Differentiate the function $y=x^3-3\log.(1+x^3)^{\frac{1}{3}}$.

Ans. $dy=\frac{3x^5dx}{1+x^3}$.

Ex. 10. Differentiate the function $y=\frac{x\ \log.x}{1-x}+\log.(1-x)$.

Ans. $dy=\frac{\log.x.dx}{(1-x)^2}$.

Ex. 11. Differentiate the function $y=x(\log.x)^n$.

Ans.

Ex. 12. Differentiate the function $y=\log.\left\{\frac{x}{a+\sqrt{a^2+x^2}}\right\}$.

Ans. $dy=\frac{adx}{x\sqrt{a^2+x^2}}$.

38. When a function consists of products and quotients of roots and powers, it is frequently advantageous to take the logarithm of each member of the equation before differentiating.

Ex. 1. Differentiate the function $y=(a+x)^m(b+x)^n$.

Taking the logarithm of each member, we have

$$\log. y=m \log.(a+x)+n \log.(b+x).$$

Hence (Art. 37) $$\frac{dy}{y}=\frac{mdx}{a+x}+\frac{ndx}{b+x};$$

therefore $$dy=(a+x)^m(b+x)^n\left(\frac{m}{a+x}+\frac{n}{b+x}\right)dx.$$

Ex. 2. Differentiate the function $y=\frac{(x+4)^2}{x+2}$.

$$Ans.\ dy=\frac{x(x+4)dx}{(x+2)^2}.$$

Ex. 3. Differentiate the function $y=\frac{x^n}{(1+x)^n}$.

$$Ans.\ dy=\frac{nx^{n-1}dx}{(1+x)^{n+1}}.$$

Ex. 4. Differentiate the function $y=\left(\frac{x-1}{x+1}\right)^{\frac{1}{2}}$.

$$Ans.\ dy=\frac{dx}{(x-1)^{\frac{1}{2}}(x+1)^{\frac{3}{2}}}.$$

Ex. 5. Differentiate the function $y=x(a^2+x^2)\sqrt{a^2-x^2}$.

$$Ans.\ dy=\frac{a^4+a^2x^2-4x^4}{(a^2-x^2)^{\frac{1}{2}}}dx.$$

Ex. 6. Differentiate the function $y=x^{\frac{3}{2}}\sqrt{\frac{1+x}{1-x}}$.

$$Ans.\ dy=\frac{3+2x-3x^2}{2(1-x)}\sqrt{\frac{x}{1-x^2}}.dx.$$

39. *Illustration from logarithmic tables.* The differential coefficient of a function denotes the limiting ratio of the increment of the function to that of the variable. It therefore denotes very nearly the actual ratio of the two increments when the increment of x is taken very small. This principle is illustrated by a table of common logarithms. What is called the

tabular difference is the increment of the logarithm corresponding to an increment of unity in the natural number. It represents, therefore, very nearly the differential of the logarithm. This will be understood from a few examples.

Ex. 1. The common logarithm of 4825 is 3.683497; what is the logarithm of 4826, assuming that the tabular difference is the differential coefficient of the logarithm?

Let x represent any number, and y its logarithm; then

$$y = \log. x.$$

Hence (Art. 35) $$dy = \frac{M dx}{x}.$$

But M, the modulus of the common system, is 0.434294; hence $$dy = \frac{0.434294}{4825} dx = 0.000090 dx.$$

The increment of the logarithm is therefore 0.000090 for an increment of unity in the corresponding natural number; hence the logarithm of 4826 is 3.683587, and this is the logarithm found in the table.

Ex. 2. Required the tabular difference for numbers between 9651 and 9652.

Ex. 3. Required the tabular difference for numbers between 5791 and 5792.

Ex. 4. Required the tabular difference for numbers between 3810 and 3811.

40. *To differentiate an exponential function.* Let a represent a constant quantity, and let

$$y = a^x.$$

Taking the logarithm of both members, we have

$$\log. y = x \log. a.$$

Differentiating both members, we have (Art. 37)

$$\frac{dy}{y} = dx \log. a.$$

Therefore $$dy = y \log. a dx = a^x \log. a dx;$$

that is, *the differential of a constant quantity raised to a*

power denoted by a variable exponent, is equal to the power multiplied by the Naperian logarithm of the constant into the differential of the exponent.

Ex. 1. Differentiate the function $y=a^{nx}$.

$$Ans.\ dy=na^{nx}\log.\ a.dx.$$

Ex. 2. Differentiate the function $y=x^n a^x$.

$$Ans.\ dy=x^{n-1}a^x(n+x\log.\ a)dx.$$

Ex. 3. Differentiate the function $y=a^x(x-1)$.

$$Ans.\ dy=a^x(x\log.\ a-\log.\ a+1)dx.$$

Ex. 4. Differentiate the function $y=\frac{a^x}{1+x}$.

$$Ans.\ dy=\frac{a^x(x\log.\ a+\log.\ a-1)dx}{(1+x)^2}.$$

Ex. 5. Differentiate the function $y=(a^x+x)^2$.

$$Ans.\ dy=2(a^x+x)(a^x\log.\ a+1)dx.$$

Ex. 6. Differentiate the function $y=\frac{a^x-1}{a^x+1}$.

$$Ans.\ dy=\frac{2a^x\log.\ a.dx}{(a^x+1)^2}.$$

41. *To differentiate a variable quantity with a variable exponent.*

Let $$u=y^x.$$

Taking the logarithm of each member, we have

$$\log.\ u=x\log.\ y.$$

Differentiating each member, we have (Arts. 24 and 37)

$$\frac{du}{u}=\log.\ ydx+\frac{xdy}{y},$$

or $$du=u\log.\ ydx+\frac{uxdy}{y}.$$

Substituting the value of u, we have

$$d(y^x)=y^x\log.\ ydx+y^x x\frac{dy}{y},$$

$$=y^x\log.\ ydx+xy^{x-1}dy;$$

that is, to differentiate a quantity formed by raising a variable quantity to a power whose exponent is variable, *differentiate*

first as though the root alone were variable, then as though the exponent alone were variable, and take the sum of the results.

Ex. 1. Differentiate the function $u=\frac{a^x}{x^x}$ or $\left(\frac{a}{x}\right)^x$.

$$Ans.\ du=\left(\frac{a}{x}\right)^x\left(\log.\frac{a}{x}-1\right)dx.$$

Ex. 2. Differentiate the function $u=x^{\frac{1}{x}}$.

$$Ans.\ du=\frac{x^{\frac{1}{x}}(1-\log. x)}{x^2}dx.$$

Ex. 3. Differentiate the function $u=\left(\frac{x}{n}\right)^{nx}$.

$$Ans.\ du=n\left(\frac{x}{n}\right)^{nx}\left\{1+\log.\frac{x}{n}\right\}dx.$$

Ex. 4. Differentiate the function $u=(\sqrt{x})^x$.

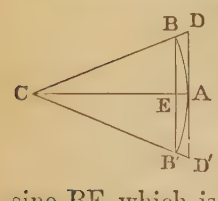
$$Ans.\ du=\frac{x^{\frac{x}{2}}}{2}(1+\log. x)dx.$$

Ex. 5. Differentiate the function $u=a^{x^x}$, in which the variable exponent is x^x.

$$Ans.\ a^{x^x}x^x(1+\log. x)\log. a dx.$$

DIFFERENTIALS OF TRIGONOMETRICAL FUNCTIONS.

42. *Arc and sine compared.* Let AB be an arc of a circle whose radius is AC. Take the arc AB′ equal to AB. Join BB′, and draw DD′ touching the circle at A. Then BE is the sine and DA the tangent of the arc AB.



The chord BB′, being a straight line, is shorter than the arc BAB′; therefore the sine BE, which is half of BB′, is less than the arc BA, which is half of BAB′. Therefore the sine is less than the arc.

Again, the area of the sector ABC is measured by

$$\tfrac{1}{2}\text{AC}\times\text{arc AB};$$

also the area of the triangle ADC is measured by

$$\tfrac{1}{2}\text{AC}\times\text{AD}.$$

But the sector ABC is less than the triangle ADC, being contained within it; hence

$$\tfrac{1}{2}\text{AC}\times\text{arc AB}<\tfrac{1}{2}\text{AC}\times\text{AD}.$$

Consequently, $$\text{arc AB}<\text{AD};$$

that is, the arc is less than its tangent; or *an arc of a circle* (not exceeding a quadrant) *is greater than its sine and less than its tangent.*

If we put h for the arc AB, we shall have

$$\text{sin. } h<h<\text{tang. } h.$$

43. *Limiting ratio of an arc to its sine.* By Trigonometry (Art. 28),

$$\frac{\text{sin. } h}{\text{tang. } h}=\frac{\text{cos. } h}{\text{R}}.$$

But the cosine of zero is equal to radius, hence the limit of the value of

$$\frac{\text{sin. } h}{\text{tang. } h} \text{ is unity;}$$

and since the arc is comprised between the sine and tangent, the limit of the value of

$$\frac{\text{sin. } h}{h} \text{ is unity;}$$

that is, *the limit of the ratio of an arc to its sine is unity.* So also *the limit of the ratio of an arc to its tangent or its chord is unity.*

44. *To find the differential of sine x.* Let $u=\text{sin. } x$.

If we increase x by h, then

$$u'=\text{sin. }(x+h),$$

and $$u'-u=\text{sin. }(x+h)-\text{sin. } x. \qquad (1)$$

By Trigonometry, Art. 75,

$$\text{sin. } a-\text{sin. } b=\frac{2}{\text{R}}\text{ sin. }\tfrac{1}{2}(a-b)\text{ cos. }\tfrac{1}{2}(a+b). \qquad (2)$$

Making $a=x+h$ and $b=x$, equation (2) becomes

$$\sin.(x+h)-\sin. x=\frac{2}{\mathrm{R}}\sin.\tfrac{1}{2}h\cos.(x+\tfrac{1}{2}h).$$

Hence equation (1) becomes

$$u'-u=\frac{2}{\mathrm{R}}\sin.\tfrac{1}{2}h\cos.(x+\tfrac{1}{2}h).$$

Dividing both members by h, and both terms of the fraction in the second member by 2, we have

$$\frac{u'-u}{h}=\frac{\sin.\frac{1}{2}h}{\frac{1}{2}h}\times\frac{\cos.(x+\frac{1}{2}h)}{\mathrm{R}};$$

that is,

$$\frac{\text{incr. } u}{\text{incr. } x}=\frac{\sin.\frac{1}{2}h}{\frac{1}{2}h}\times\frac{\cos.(x+\frac{1}{2}h)}{\mathrm{R}}$$

Now at the limit (Art. 43), $\dfrac{\sin.\frac{1}{2}h}{\frac{1}{2}h}=1.$

Hence,

$$\frac{du}{dx}=\frac{\cos. x}{\mathrm{R}},$$

or

$$du=d\sin. x=\frac{\cos. x\,dx}{\mathrm{R}}.$$

Radius is usually taken as unity, in which case we have

$$d\sin. x=\cos. x\,dx.$$

45. *To find the differential of cosine x.* Let $u=\cos. x$.

Then

$$du=d\cos. x=d\sin.(90°-x). \qquad (1)$$

But by Art. 44,

$$d\sin.(90°-x)=\frac{\cos.(90°-x)\,d\,(90°-x)}{\mathrm{R}}$$

But

$$\cos.(90°-x)=\sin. x,$$

and

$$d\,(90°-x)=-dx.$$

Hence by substitution equation (1) becomes

$$d\cos. x=-\frac{\sin. x\,dx}{\mathrm{R}}.$$

Also, since the versed sine of an arc, less than ninety degrees, is equal to radius minus the cosine, we have

$$d\text{ versed sin. }x=d\,(\mathrm{R}-\cos. x)=\frac{\sin. x\,dx}{\mathrm{R}}.$$

46. *To find the differential of tangent x.* Let $u = \text{tang. } x$.

Since $\text{tang. } x = \dfrac{\text{R sin. } x}{\text{cos. } x}$ (Trig., Art. 28), we have

$$d \text{ tang. } x = \frac{\text{R cos. } x\, d \text{ sin. } x - \text{R sin. } x\, d \text{ cos. } x}{\text{cos.}^2 x},$$

$$= \frac{(\text{cos.}^2 x + \text{sin.}^2 x)\, dx}{\text{cos.}^2 x}.$$

But $$\text{cos.}^2 x + \text{sin.}^2 x = \text{R}^2.$$

Hence $$d \text{ tang. } x = \frac{\text{R}^2 dx}{\text{cos.}^2 x} = \frac{\text{sec.}^2 x dx}{\text{R}^2}.$$

47. *To find the differential of cotangent x.* Let $u = \text{cot. } x$.

Then $$du = d \text{ cot. } x = d \text{ tang. } (90° - x). \qquad (1)$$

But by Art. 46,

$$d \text{ tang. } (90° - x) = \frac{\text{R}^2 d(90° - x)}{\text{cos.}^2 (90° - x)}.$$

Also, $$d(90° - x) = -dx,$$

and $$\text{cos.}^2 (90° - x) = \text{sin.}^2 x.$$

Hence by substitution equation (1) becomes

$$d \text{ cot. } x = -\frac{\text{R}^2 dx}{\text{sin.}^2 x} = -\frac{\text{cosec.}^2 x dx}{\text{R}^2}.$$

48. *To find the differential of secant x.* Let $u = \text{sec. } x$.

Since $\text{sec. } x = \dfrac{\text{R}^2}{\text{cos. } x}$, we have

$$d \text{ sec. } x = -\frac{\text{R}^2\, d \text{ cos. } x}{\text{cos.}^2 x} = \frac{\text{R sin. } x dx}{\text{cos.}^2 x} = \frac{\text{tang. } x \text{ sec. } x dx}{\text{R}^2}.$$

49. *To find the differential of cosecant x.* Let $u = \text{cosec. } x$.

Then $$du = d \text{ cosec. } x = d \text{ sec. } (90° - x).$$

But $$d \text{ sec. } (90° - x) = \frac{\text{cot. } x \text{ cosec. } x d(90° - x)}{\text{R}^2}.$$

Hence $$d \text{ cosec. } x = -\frac{\text{cot. } x \text{ cosec. } x dx}{\text{R}^2}.$$

Ex. 1. In what part of the quadrant does the arc increase twice as fast as its sine?

Let $y = \sin. x$, then $dy = \cos. x dx$; that is, the rate of increase of the sine is equal to the rate of increase of the arc multiplied by cos. x. Now $\cos. 60° = \frac{1}{2}$.

Hence at 60° the arc is increasing twice as fast as its sine.

Ex. 2. In what part of the quadrant does the arc increase three times as fast as its sine? *Ans.* At 70° 32′.

Ex. 3. In what part of the quadrant does the arc increase half as fast as its tangent? *Ans.* At 45°.

Ex. 4. The natural sine of 30° is 0.500000; what is the sine of 30° 1′?

The length of an arc of 1′ is found by dividing 3.14159 by the number of minutes in 180°, and we obtain 0.0002909, which represents the increment of the arc. The increment of its sine will be found by multiplying this number by cos. 30°, or 0.866, and we obtain 0.000252, which is the tabular difference corresponding to the natural sine of 30°. Hence the sine of 30° 1′ is 0.500252.

Ex. 5. The natural sine of 10° 30′ is 0.182236; what is the sine of 10° 31′? *Ans.* 0.182522.

Ex. 6. The natural cosine of 65° 10′ is 0.419980; what is the cosine of 65° 11′? *Ans.* 0.419716.

Ex. 7. The natural tangent of 45° is 1.000000; what is the tangent of 45° 1′? *Ans.* 1.000582.

Ex. 8. The natural cotangent of 60° is 0.577350; what is the cotangent of 60° 1′? *Ans.* 0.576962.

Ex. 9. Differentiate the function $y = \sin. mx$.

Ans. $dy = \cos. mx d(mx) = m \cos. mx dx$.

Ex. 10. Differentiate the function $y = \sin.^3 x$.

Ans. $dy = 3 \sin.^2 x d \sin. x = 3 \sin.^2 x \cos. x dx$.

Ex. 11. Differentiate the function $y = \sin.^n x$.

Ans. $n \sin.^{n-1} x \cos. x dx$.

Ex. 12. Differentiate the function $y = a^{\sin. x}$.

Ans. $dy = a^{\sin. x} \log. a \cos. x dx$.

Ex. 13. Differentiate the function $y=(\cos. x)^n$.

$$Ans.\ dy=-n(\cos. x)^{n-1}\sin. x dx.$$

Ex. 14. Differentiate the function $y=\sin. 2x \cos. x$.

$$Ans.\ \frac{dy}{dx}=2\cos. 2x \cos. x-\sin. x \sin. 2x.$$

$$=\cos. 2x \cos. x+\cos. 3x.$$

Ex. 15. Differentiate the function $y=x-\sin. x \cos. x$.

$$Ans.\ dy=2\sin.^2 x dx.$$

Ex. 16. Differentiate the function $y=\text{tang.}\, x-x$.

$$Ans.\ dy=\text{tang.}^2 x dx.$$

Ex. 17. Differentiate the function $y=\text{tang.}^n x$.

$$Ans.\ dy=\frac{n\,\text{tang.}^{n-1} x dx}{\cos.^2 x}.$$

Ex. 18. Differentiate the function $y=\sqrt{\text{tang. } 2x}$.

$$Ans.\ dy=\frac{\sec.^2 2x dx}{\sqrt{\text{tang. } 2x}}.$$

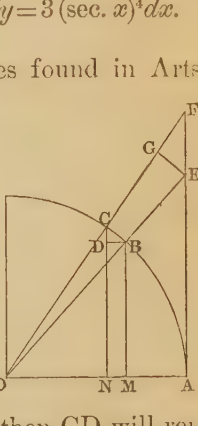
Ex. 19. Differentiate the function $y=\cot.^2 x$.

$$Ans.\ dy=-\frac{2\cot. x dx}{\sin.^2 x}.$$

Ex. 20. Differentiate the function $y=3\,\text{tang.}\, x+(\text{tang.}\, x)^3$.

$$Ans.\ dy=3(\sec. x)^4 dx.$$

50. *Geometrical illustration.* The values found in Arts. 44–49 for the differentials of sin. x, cos. x, etc., may be deduced from the principles of elementary geometry.



Let AB be an arc of a circle whose centre is O; let BM be its sine, OM its cosine, AE its tangent, and OE its secant, and let the radius OA be taken as unity. Let the arc AB be represented by x, and suppose x to receive a small increment BC. Draw BD perpendicular to CN, and with radius OE draw the circular arc EG; then CD will represent the increment of sin. x, BD taken negatively will represent the increment of cos. x, EF will represent the increment

of tang. x, and GF the increment of sec. x. Suppose the chords BC and EG to be drawn, forming the rectilinear triangles BCD and EFG, then, when the arc BC is indefinitely diminished, since the limiting value of the angle OBC is 90°, the limit of the ratio of OBC to MBD is unity; that is, the limit of the ratio of DBC to MBO is unity. So also the limit of the ratio of BCD to BOM is unity; that is, the triangle BCD in its limiting form is similar to the triangle BOM or EOA. For a like reason, the triangle EGF in its limiting form is similar to the triangle OMB or OAE. Hence we have the following equations:

The limit of $\dfrac{\text{CD}}{\text{CB}}$ or $\dfrac{d \sin. x}{dx}$ is $\dfrac{\text{OM}}{\text{OB}}$ or cos. x.

The limit of $\dfrac{\text{BD}}{\text{CB}}$ or $\dfrac{d \cos. x}{dx}$ is $\dfrac{\text{BM}}{\text{OB}}$ or sin. x.

Also, by similar figures,

$$\frac{\text{EG}}{\text{BC}} = \frac{\text{EO}}{\text{BO}} = \frac{\text{EO}}{\text{AO}} = \text{sec. } x;$$

and the limit of $\dfrac{\text{EF}}{\text{EG}}$ is $\dfrac{\text{EO}}{\text{AO}}$ or sec. x.

Hence the limit of

$$\frac{\text{EF}}{\text{BC}}, \text{ that is, of } \frac{\text{EF}}{\text{EG}} \times \frac{\text{EG}}{\text{BC}} \text{ or } \frac{d \text{ tang. } x}{dx} = \text{sec.}^2 x.$$

So also the limit of $\dfrac{\text{FG}}{\text{EG}}$ is $\dfrac{\text{AE}}{\text{AO}}$ or tang. x.

Hence the limit of

$$\frac{\text{FG}}{\text{BC}}, \text{ that is, of } \frac{\text{FG}}{\text{EG}} \times \frac{\text{EG}}{\text{BC}} \text{ or } \frac{d \text{ sec. } x}{dx} = \text{tang. } x \text{ sec. } x.$$

51. *Differentials of logarithmic sine, cosine, etc.* By combining the results of Arts. 44–49 with the rule in Art. 36, we may differentiate the logarithm of any trigonometrical function.

By Art. 36, calling radius unity, we have

1. $d \log. \sin. x = \frac{M d \sin. x}{\sin. x} = \frac{M \cos. x dx}{\sin. x} = M \cot. x dx.$

2. $d \log. \cos. x = \frac{M d \cos. x}{\cos. x} = -\frac{M \sin. x dx}{\cos. x} = -M \tang. x dx.$

3. $d \log. \tang. x = \frac{M d \tang. x}{\tang. x} = \frac{M dx}{\cos.^2 x \tang. x} = \frac{M dx}{\sin. x \cos. x}$
$= \frac{2M dx}{\sin. 2x}.$

4. $d \log. \cot. x = \frac{M d \cot. x}{\cot. x} = -\frac{M dx}{\sin.^2 x \cot. x} = -\frac{M dx}{\sin. x \cos. x}$
$= -\frac{2M dx}{\sin. 2x}.$

Hence the differentials of the logarithmic tangent and cotangent of any arc differ only in sign.

It may also be easily proved that the differential of the logarithmic tangent is equal to the arithmetical sum of the differentials of the sine and cosine of the same arc.

Ex. 1. In what part of the quadrant does the arc increase at the same rate as its logarithmic sine?

We have found $d \log. \sin. x = M \cot. x dx$;
that is, the rate of increase of the logarithmic sine is equal to the rate of increase of the arc multiplied by M cot. x, or $\frac{M}{\tang. x}$. Now $M = 0.434294$, and $\tang. 23° 28' = 0.434294$. Hence at $23° 28'$ the arc and its logarithmic sine are increasing at the same rate.

Ex. 2. In what part of the quadrant does the arc increase at the same rate as its logarithmic tangent? *Ans.* $30° 9'$.

Ex. 3. The logarithmic sine of $30° 5'$ is 9.700062; what is the logarithmic sine of $30° 6'$?

The increment of x found in Ex. 4, Art. 49 is 0.0002909, which, multiplied by M, gives $M dx = 0.0001263$.

Multiply this result by cot. $30° 5'$, and we have $d \log. \sin. x = 0.000218$.

Hence the logarithmic sine of 30° 6′ is 9.700280.

Ex. 4. The logarithmic cosine of 45° 35′ is 9.845018; what is the logarithmic cosine of 45° 36′? *Ans.* 9.844889.

Ex. 5. The logarithmic tangent of 46° 30′ is 10.022750; what is the logarithmic tangent of 46° 31′?

Ans. 10.023003.

Ex. 6. The logarithmic cotangent of 50° 15′ is 9.919962; what is the logarithmic cotangent of 50° 16′? *Ans.* 9.919705.

In the following examples Naperian logarithms are supposed to be used:

Ex. 7. Differentiate the function $y=\log.\sin.2x$.

$$Ans.\ dy=\frac{d\sin.2x}{\sin.2x}=\frac{2\cos.2x dx}{\sin.2x}=2\cot.2x dx.$$

Ex. 8. Differentiate the function $y=\log.\sin.^2x$.

$$Ans.\ dy=2\cot.x dx.$$

Ex. 9. Differentiate the function

$$y=\log.\sqrt{\frac{1+\sin.x}{1-\sin.x}}=\tfrac{1}{2}\log.(1+\sin.x)-\tfrac{1}{2}\log.(1-\sin.x).$$

$$Ans.\ dy=\tfrac{1}{2}\frac{d(1+\sin.x)}{1+\sin.x}-\tfrac{1}{2}\frac{d(1-\sin.x)}{1-\sin.x}=\frac{dx}{\cos.x}.$$

Ex. 10. Differentiate the function $y=\log.\sqrt{\sin.x}$.

$$Ans.\ dy=\tfrac{1}{2}\cot.x dx.$$

Ex. 11. Differentiate the function $y=\log.\sqrt{\frac{1-\cos.x}{1+\cos.x}}$.

$$Ans.\ dy=\frac{dx}{\sin.x}.$$

Ex. 12. Differentiate the function $y=\log\sqrt{\sin.x}+\log.\sqrt{\cos.x}$.

$$Ans.\ dy=\frac{dx}{\text{tang.}\,2x}.$$

INVERSE TRIGONOMETRICAL FUNCTIONS.

52. *Notation adopted.* In the preceding articles we have found the differentials of the sine, cosine, etc., in terms of the arc as an independent variable. It is sometimes more convenient to regard the arc as the function, and the sine, cosine,

etc., as the independent variable. Such functions are called *inverse trigonometrical functions*, and they are commonly designated by the symbol -1, written as an exponent. Thus the expression $\sin.^{-1}x$ denotes the arc whose sine is x. Hence, if $y=\sin.^{-1}x$, we have $\sin. y=x$. So also $\cos.^{-1}x$, $\tang.^{-1}x$, etc., denote the arc whose cosine is x, the arc whose tangent is x, etc. Inverse trigonometrical functions are sometimes called *circular* functions.

53. *Differentiation of inverse trigonometrical functions.* Formulas for the differentiation of inverse trigonometrical functions may be deduced from the formulas of Arts. 44–49.

1. To differentiate $y=\sin.^{-1}x$.

Here the direct function is $\sin. y=x$. Hence $d \sin. y=dx$;

that is, $\cos. y dy=dx$, or $dy=\dfrac{dx}{\cos. y}$.

But since $\sin. y=x$, $\cos. y=\sqrt{1-x^2}$.

Hence $$dy=\frac{dx}{\sqrt{1-x^2}},$$ radius being unity.

2. To differentiate $y=\cos.^{-1}x$.

Here $$\cos. y=x.$$

Hence $$-\sin. y dy=dx,$$

or $$dy=\frac{-dx}{\sin. y}=\frac{-dx}{\sqrt{1-x^2}}.$$

3. To differentiate $y=\text{vers.}^{-1}x$.

Here $$\text{vers. } y=x.$$

Hence $$\sin. y dy=dx.$$

But $$\sin. y=\sqrt{1-\cos.^2 y}=\sqrt{1-(1-\text{vers. } y)^2}.$$
$$=\sqrt{1-(1-x)^2}=\sqrt{2x-x^2}.$$

Hence $$dy=\frac{dx}{\sqrt{2x-x^2}}.$$

4. To differentiate $y=\tang.^{-1}x$.

Here $$\tang. y=x.$$

Hence $$\sec.^2 y dy=dx,$$

or $$dy=\frac{dx}{\sec.^2 y}=\frac{dx}{1+x^2}.$$

5. To differentiate $y=\text{cot.}^{-1}x$.

Here $$\text{cot. } y=x.$$

Hence $$-\text{cosec.}^2 y\,dy=dx;$$

$$dy=\frac{-dx}{\text{cosec.}^2 y}=\frac{-dx}{1+x^2}.$$

6. To differentiate $y=\text{sec.}^{-1}x$.

Here $$\text{sec. } y=x;$$

$$\text{tang. } y \text{ sec. } y\,dy=dx;$$

$$dy=\frac{dx}{\text{tang. } y \text{ sec. } y}=\frac{dx}{x\sqrt{x^2-1}}.$$

7. To differentiate $y=\text{cosec.}^{-1}x$.

Here $$\text{cosec. } y=x;$$

$$-\text{cot. } y \text{ cosec. } y\,dy=dx;$$

$$dy=\frac{-dx}{\text{cot. } y \text{ cosec. } y}=\frac{-dx}{x\sqrt{x^2-1}}.$$

Ex. 1. Differentiate the function $y=\text{sin.}^{-1}\frac{x}{\sqrt{1+x^2}}$.

Here $$\text{sin. } y=\frac{x}{\sqrt{1+x^2}}.$$

Differentiating, we find $$\text{cos. } y\,dy=\frac{dx}{(1+x^2)^{\frac{3}{2}}}.$$

Hence $$dy=\frac{1}{\text{cos. } y}\times\frac{dx}{(1+x^2)^{\frac{3}{2}}}.$$

But $$\text{cos. } y=\sqrt{1-\text{sin.}^2 y}=\sqrt{1-\frac{x^2}{1+x^2}}=\frac{1}{(1+x^2)^{\frac{1}{2}}}.$$

Hence, by substitution,

$$dy=(1+x^2)^{\frac{1}{2}}\times\frac{dx}{(1+x^2)^{\frac{3}{2}}}=\frac{dx}{1+x^2}.$$

Ex. 2. Differentiate the function $y=\text{sin.}^{-1}\frac{1-x^2}{1+x^2}$.

$$Ans.\ dy=-\frac{2dx}{1+x^2}.$$

Ex. 3. Differentiate the function $y=\text{cos.}^{-1}(4x^3-3x)$.

Here $$\cos. y = 4x^3 - 3x.$$

But by Trig., Art. 79, $\cos. 3b = 4 \cos.^3 b - 3 \cos. b$,

or $$\cos. y = 4 \cos.^3\left(\frac{y}{3}\right) - 3 \cos.\left(\frac{y}{3}\right).$$

Hence $$x = \cos.\left(\frac{y}{3}\right).$$

$$Ans.\ dy = \frac{-3dx}{\sqrt{1-x^2}}.$$

Ex. 4. Differentiate the function $y = \sin.^{-1}(3x - 4x^3)$.

$$Ans.\ dy = \frac{3dx}{\sqrt{1-x^2}}.$$

Ex. 5. Differentiate the function $y = \text{tang.}^{-1}\frac{2x}{1-x^2}$.

$$Ans.\ dy = \frac{2dx}{1+x^2}.$$

Ex. 6. Differentiate the function $y = \sin.^{-1}(2x - 1)$.

$$Ans.\ dy = \frac{dx}{\sqrt{x-x^2}}.$$

Ex. 7. Differentiate the function $y = \sin.^{-1} mx$.

$$Ans.\ dy = \frac{mdx}{\sqrt{1-m^2x^2}}.$$

Ex. 8. Differentiate the function $y = \text{tang.}^{-1}\frac{x}{2}$.

$$Ans.\ dy = \frac{2dx}{4+x^2}.$$

Ex. 9. Differentiate the function $y = \cos.^{-1}\frac{1}{\sqrt{1+x^2}}$.

$$Ans.\ dy = \frac{dx}{1+x^2}.$$

Ex. 10. Differentiate the function $y = \text{tang.}^{-1}\frac{x}{\sqrt{1-x^2}}$.

$$Ans.\ dy = \frac{dx}{\sqrt{1-x^2}}.$$

CHAPTER III.

SUCCESSIVE DIFFERENTIATION. EXPANSION OF FUNCTIONS IN SERIES. FUNCTIONS OF SEVERAL INDEPENDENT VARIABLES. VANISHING FRACTIONS.

54. *Second differential coefficient.* The differential coefficient derived from any function of a variable is either a constant quantity or a new function of the same variable. This function may therefore be differentiated in the same manner as the original function; and the differential coefficient of this second function is called the *second differential coefficient* of the original function.

Thus, taking the function $u = ax^4$, we have for the first differential coefficient,

$$\frac{du}{dx} = 4ax^3.$$

This coefficient is a function of x different from the original function. If we differentiate each member of the last equation, we have

$$d\left(\frac{du}{dx}\right) = 12ax^2dx.$$

But since dx is supposed to be a constant, we have

$$d\left(\frac{du}{dx}\right) = \frac{d(du)}{dx} = \frac{d.du}{dx}, \text{ which is written } \frac{d^2u}{dx}.$$

The symbol d^2u denotes the differential of the differential of u, and is read *second differential of* u.

Hence, we have

$$\frac{d^2u}{dx} = 12ax^2dx;$$

or, dividing each side by dx,

$$\frac{d^2u}{dx^2} = 12ax^2.$$

where dx^2 denotes the square of dx, and is the same as if written $(dx)^2$.

The expression $12ax^2$ being the differential coefficient of the first differential coefficient, is called the *second differential coefficient.*

Since the second differential coefficient is a function of x, we may differentiate it as a new function, and we obtain

$$\frac{d^3u}{dx^2}=24axdx;$$

or, dividing each side by dx, we have

$$\frac{d^3u}{dx^3}=24ax,$$

which is the differential coefficient of the second differential coefficient, and is called the *third differential coefficient.*

The third differential coefficient $\frac{d^3u}{dx^3}$ is read third differential of u divided by the cube of the differential of x.

In the same way the fourth differential coefficient may be derived from the third, and so on.

The symbol d^2u must be carefully distinguished from du^2, which denotes the second power of du; and also from $d(u^2)$, which denotes the differential of the second power of u.

EXAMPLES.

Find the successive differential coefficients of the following functions:

1. $u=ax^n$.

$$Ans.\ \frac{du}{dx}=nax^{n-1};\ \frac{d^2u}{dx^2}=n(n-1)ax^{n-2};$$

$$\frac{d^3u}{dx^3}=n(n-1)(n-2)ax^{n-3};\ \text{etc.}$$

It will be seen that by each differentiation the exponent of the power is diminished by unity. Hence, when n is a positive whole number, the exponent will finally be reduced to zero, and the corresponding differential coefficient will be a constant; the next differential coefficient will therefore be zero, and the number of successive differential coefficients will be finite. But if the exponent be fractional or negative, the re-

sult can never become zero by the continued subtraction of unity; hence the successive differential coefficients will always contain x, and the number of differential coefficients will be infinite.

2. $u=ax^3+bx^2$.

$$Ans.\ \frac{du}{dx}=3ax^2+2bx;\ \frac{d^2u}{dx^2}=6ax+2b;\ \frac{d^3u}{dx^3}=6a.$$

3. $u=x^4+x^3+x^2+x+1$.

$$Ans.\ \frac{du}{dx}=4x^3+3x^2+2x+1;\ \frac{d^2u}{dx^2}=12x^2+6x+2;$$
$$\frac{d^3u}{dx^3}=24x+6;\ \frac{d^4u}{dx^4}=24.$$

4. $u=6x^4-4x^3-5x^2$.

$$Ans.\ \frac{du}{dx}=24x^3-12x^2-10x;\ \frac{d^2u}{dx^2}=72x^2-24x-10;$$
$$\frac{d^3u}{dx^3}=144x-24;\ \frac{d^4u}{dx^4}=144.$$

5. $u=x^{-2}$.

$$Ans.\ \frac{du}{dx}=-2x^{-3};\ \frac{d^2u}{dx^2}=6x^{-4};\ \frac{d^3u}{dx^3}=-24x^{-5};$$
$$\frac{d^4u}{dx^4}=120x^{-6}.\ \text{etc.}$$

6. $u=ax^{\frac{1}{2}}$.

$$Ans.\ \frac{du}{dx}=\tfrac{1}{2}ax^{-\frac{1}{2}};\ \frac{d^2u}{dx^2}=-\tfrac{1}{4}ax^{-\frac{3}{2}};\ \frac{d^3u}{dx^3}=\tfrac{3}{8}ax^{-\frac{5}{2}};$$
$$\frac{d^4u}{dx^4}=-\tfrac{15}{16}ax^{-\frac{7}{2}};\ \text{etc.}$$

7. $u=\sin.\,x$.

$$Ans.\ \frac{du}{dx}=\cos.\,x;\ \frac{d^2u}{dx^2}=-\sin.\,x;\ \frac{d^3u}{dx^3}=-\cos.\,x;$$
$$\frac{d^4u}{dx^4}=\sin.\,x;\ \text{etc.}$$

8. $u=\log.(x+1)$.

$$Ans.\ \frac{du}{dx}=(x+1)^{-1};\ \frac{d^2u}{dx^2}=-(x+1)^{-2};\ \frac{d^3u}{dx^3}=2(x+1)^{-3};$$
$$\frac{d^4u}{dx^4}=-6(x+1)^{-4};\ \text{etc.}$$

9. $u=a^x$.

$$Ans.\ \frac{du}{dx}=a^x \log.\, a;\ \frac{d^2u}{dx^2}=a^x(\log.\, a)^2;\ \text{etc.};$$

$$\frac{d^nu}{dx^n}=a^x(\log.\, a)^n.$$

10. $u=\dfrac{1}{1+x^2}$.

$$Ans.\ \frac{du}{dx}=\frac{-2x}{(1+x^2)^2};\ \frac{d^2u}{dx^2}=\frac{6x^2-2}{(1+x^2)^3};\ \frac{d^3u}{dx^3}=\frac{24x(1-x^2)}{(1+x^2)^4};\ \text{etc.}$$

55. There are four leading applications of the differential calculus in which integral calculus does not enter, viz.: 1. Expansion of functions in series; 2. Limiting values of functions which assume an indeterminate form; 3. Maxima and minima values of functions; and 4. Theory of curve lines and curve surfaces. We shall briefly consider each of these applications.

MACLAURIN'S THEOREM.

56. The object of Maclaurin's theorem is to explain the method of expanding a function of a single variable into a series of terms arranged according to the ascending powers of that variable with constant coefficients.

Let u represent any function of x; and if the function can be expanded in a series of positive integral powers of x, let

$$u=A+Bx+Cx^2+Dx^3+Ex^4+\ \text{etc.}, \qquad (1)$$

in which A, B, C, etc., are independent of x, and depend upon the constants which enter into the given function. It is required to determine such values for the constants A, B, C, etc., as will cause the assumed development to be true for all values of x.

Differentiating, we obtain the successive differential coefficients of u,

$$\frac{du}{dx}=B+2Cx+3Dx^2+4Ex^3+\ \text{etc.} \qquad (2)$$

$$\frac{d^2u}{dx^2}=2C+2.3Dx+3.4Ex^2+\ \text{etc.}; \qquad (3)$$

$$\frac{d^3u}{dx^3}=2.3\mathrm{D}+2.3.4\mathrm{E}x+\text{ etc.}; \qquad (4)$$

etc., etc.

But by hypothesis, the coefficients A, B, C, etc., are independent of x; they must therefore remain the same whatever value may be ascribed to x; and if we can determine them for one value of x, we shall have their values in all cases. In equations 1, 2, 3, 4, etc., let us make $x=0$; and let (u) denote what u becomes under this hypothesis; let $\left(\frac{du}{dx}\right)$ denote what $\frac{du}{dx}$ becomes; $\left(\frac{d^2u}{dx^2}\right)$ denote what $\frac{d^2u}{dx^2}$ becomes, and so on; the preceding equations furnish us

$$(u)=\mathrm{A}, \qquad \text{therefore} \qquad \mathrm{A}=(u);$$

$$\left(\frac{du}{dx}\right)=\mathrm{B}, \qquad \text{“} \qquad \mathrm{B}=\left(\frac{du}{dx}\right);$$

$$\left(\frac{d^2u}{dx^2}\right)=2\mathrm{C}, \qquad \text{“} \qquad \mathrm{C}=\frac{1}{2}\left(\frac{d^2u}{dx^2}\right);$$

$$\left(\frac{d^3u}{dx^3}\right)=2.3\mathrm{D}, \qquad \text{“} \qquad \mathrm{D}=\frac{1}{2.3}\left(\frac{d^3u}{dx^3}\right).$$

Substituting these values of A, B, C, etc., in equation (1), it becomes

$$u=(u)+\left(\frac{du}{dx}\right)\frac{x}{1}+\left(\frac{d^2u}{dx^2}\right)\frac{x^2}{1.2}+\left(\frac{d^3u}{dx^3}\right)\frac{x^3}{1.2.3}+\text{ etc.}, \qquad (5)$$

which is known by the name of *Maclaurin's theorem.*

It will thus be seen that Maclaurin's theorem develops a function of one variable into a series of terms of which the first is the value of the given function when $x=0$; the second term is x multiplied by the value of the first differential coefficient when $x=0$; the third term is $\frac{x^2}{2}$ multiplied by the value of the second differential coefficient when $x=0$; etc.

Ex. 1. Expand $u=(a+x)^6$ by Maclaurin's theorem.

Differentiating, we have

$$\frac{du}{dx}=6(a+x)^5, \qquad \text{hence } \left(\frac{du}{dx}\right)=6a^5;$$

$$\frac{d^2u}{dx^2}=5.6(a+x)^4, \qquad \text{“ } \left(\frac{d^2u}{dx^2}\right)=5.6a^4;$$

$$\frac{d^3u}{dx^3}=4.5.6(a+x)^3, \qquad \text{“ } \left(\frac{d^3u}{dx^3}\right)=4.5.6a^3;$$

$$\frac{d^4u}{dx^4}=3.4.5.6(a+x)^2, \qquad \text{“ } \left(\frac{d^4u}{dx^4}\right)=3.4.5.6a^2;$$

$$\frac{d^5u}{dx^5}=2.3.4.5.6(a+x), \qquad \text{“ } \left(\frac{d^5u}{dx^5}\right)=2.3.4.5.6a;$$

$$\frac{d^6u}{dx^6}=1.2.3.4.5.6, \qquad \text{“ } \left(\frac{d^6u}{dx^6}\right)=1.2.3.4.5.6.$$

Here the differentiation terminates. Substituting these values in Maclaurin's theorem, we obtain

$$u=(a+x)^6=a^6+6a^5x+\frac{5.6}{2}a^4x^2+\frac{4.5.6}{2.3}a^3x^3+\frac{3.4.5.6}{2.3.4}a^2x^4$$
$$+\frac{2.3.4.5.6}{2.3.4.5}ax^5+\frac{1.2.3.4.5.6x^6}{2.3.4.5.6};$$

or, reducing, we have $u=(a+x)^6=a^6+6a^5x+15a^4x^2+20a^3x^3+15a^2x^4+6ax^5+x^6$, the same result as would be obtained by actual multiplication, or by the binomial theorem.

Ex. 2. Expand $u=(a+x)^n$ by Maclaurin's theorem.

$$\frac{du}{dx}=n(a+x)^{n-1}, \qquad \text{hence } \left(\frac{du}{dx}\right)=na^{n-1};$$

$$\frac{d^2u}{dx^2}=n(n-1)(a+x)^{n-2}, \text{ “ } \left(\frac{d^2u}{dx^2}\right)=n(n-1)a^{n-2};$$

etc., etc.

Substituting these values in Maclaurin's theorem, we have

$$u \text{ or } (a+x)^n=a^n+\frac{n}{1}a^{n-1}x+\frac{n(n-1)}{1.2}a^{n-2}x^2+\text{ etc.},$$

which is the binomial theorem.

Ex. 3. Expand $u=\frac{1}{a+x}=(a+x)^{-1}$.

Ans. $u=a^{-1}-a^{-2}x+a^{-3}x^2-a^{-4}x^3+$ etc.

Ex. 4. Expand $u=(a-x)^{-2}$.

Ans. $u=a^{-2}+2a^{-3}x+3a^{-4}x^2+4a^{-5}x^3+5a^{-6}x^4+$ etc.

Ex. 5. Expand $u=(a+x)^{-3}$.

Ans. $u=a^{-3}-3a^{-4}x+6a^{-5}x^2-10a^{-6}x^3+15a^{-7}x^4-$ etc.

Ex. 6. Expand $u=(a+x)^{\frac{1}{2}}$.

Ans. $u=a^{\frac{1}{2}}+\frac{1}{2}a^{-\frac{1}{2}}x-\frac{1}{2.4}a^{-\frac{3}{2}}x^2+\frac{1.3}{2.4.6}a^{-\frac{5}{2}}x^3-$ etc.

Ex. 7. Expand $u=(a^2+bx)^{\frac{1}{2}}$.

Ans. $u=a+\frac{a^{-1}bx}{2}-\frac{a^{-3}b^2x^2}{8}+\frac{a^{-5}b^3x^3}{16}-$ etc.

Ex. 8. Expand $u=(1+x^2)^{\frac{1}{2}}$.

Ans. $u=1+\frac{x^2}{2}-\frac{x^4}{8}+\frac{x^6}{16}-\frac{5x^8}{128}+$ etc.

Ex. 9. Expand $u=\sin. x$.

Ans. $u=x-\frac{x^3}{2.3}+\frac{x^5}{2.3.4.5}-\frac{x^7}{2.3.4.5.6.7}+$ etc.

Ex. 10. Expand $u=\cos. x$.

Ans. $u=1-\frac{x^2}{2}+\frac{x^4}{2.3.4}-\frac{x^6}{2.3.4.5.6}+$ etc.

Ex. 11. Expand $u=\log.(1+x)$.

Ans. $u=\mathrm{M}(x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+$ etc.).

Ex. 12. Expand $u=a^x$.

The successive differential coefficients have already been given in Ex. 9, Art. 54. For convenience, we will represent log. a by k. We shall then have

$$(u)=1\,;\ \left(\frac{du}{dx}\right)=k\,;\ \left(\frac{d^2u}{dx^2}\right)=k^2\,;\ \left(\frac{d^3u}{dx^3}\right)=k^3, \text{ etc.}$$

Hence, by substitution,

$$a^x=1+kx+\frac{k^2x^2}{1.2}+\frac{k^3x^3}{1.2.3}+\text{ etc.}$$

which is the *exponential theorem.*

If $a=e$, the Naperian base, we have

$$a^x=e^x=1+x+\frac{x^2}{1.2}+\frac{x^3}{1.2.3}+\frac{x^4}{1.2.3.4}+\text{ etc.}$$

If $x=1$, we have

$$e=1+1+\frac{1}{1.2}+\frac{1}{1.2.3}+\frac{1}{1.2.3.4}+\text{ etc.},$$

a series which gives the value of e the base of the Naperian logarithms.

57. *Arc in terms of its tangent.* By means of Maclaurin's theorem, we can develop the arc in terms of its sine, tangent, etc., and hence obtain the value of the circumference of a circle of any given radius.

Let $u=\text{tang.}^{-1}x$. Hence $(u)=0$.

By Art. 53, (4), $\frac{du}{dx}=\frac{1}{1+x^2}$, which, by division, becomes

$$1-x^2+x^4-x^6+\text{ etc.} \quad \text{Hence } \left(\frac{du}{dx}\right)=1.$$

By successive differentiation, we have

$$\frac{d^2u}{dx^2}=-2x+4x^3-6x^5+\text{ etc.}, \quad \text{hence } \left(\frac{d^2u}{dx^2}\right)=0;$$

$$\frac{d^3u}{dx^3}=-2+3.4x^2-5.6x^4+\text{ etc.}, \quad \text{“} \quad \left(\frac{d^3u}{dx^3}\right)=-2;$$

$$\frac{d^4u}{dx^4}=2.3.4x-4.5.6x^3+\text{ etc.}, \quad \text{“} \quad \left(\frac{d^4u}{dx^4}\right)=0;$$

$$\frac{d^5u}{dx^5}=2.3.4-3.4.5.6x^2+\text{ etc.}, \quad \text{“} \quad \left(\frac{d^5u}{dx^5}\right)=2.3.4;$$

etc., etc.

Substituting these values in Maclaurin's theorem, we have

$$u=x-\frac{2x^3}{2.3}+\frac{2.3.4x^5}{2.3.4.5}-\text{ etc.},$$

or

$$u=\text{tang.}^{-1}x=\frac{x}{1}-\frac{x^3}{3}+\frac{x^5}{5}-\text{ etc.}, \qquad (1)$$

which is an expression for the length of an arc in terms of its tangent. From this development we are enabled to find the value of π. Thus, let $u=45^\circ=\frac{\pi}{4}$, whence $x=1$, and we have

$$\text{arc } 45^\circ=1-\tfrac{1}{3}+\tfrac{1}{5}-\tfrac{1}{7}+\tfrac{1}{9}-\tfrac{1}{11}+\tfrac{1}{13}-\text{ etc.}$$

From this series an approximation may be made to the circumference of a circle, but the series converges so slowly that an accurate result can not be obtained from it without very great labor.

58. *Computation of the value of* π. In order that series (1), Art. 57, may converge rapidly, x must be a small fraction. This object may be accomplished by employing the principle that *the arc* $45°$ *is equal to the arc whose tangent is* $\frac{1}{2}$ *plus the arc whose tangent is* $\frac{1}{3}$.

This principle may be proved as follows. From Trig., Art. 77,

$$\text{tang.}\,(a+b)=\frac{\text{tang.}\,a+\text{tang.}\,b}{1-\text{tang.}\,a\ \text{tang.}\,b}.$$

Assume $a+b=45°$, and tang. $a=\frac{1}{2}$.

Then $\text{tang.}\,45°=1=\frac{\frac{1}{2}+\text{tang.}\,b}{1-\frac{1}{2}\,\text{tang.}\,b}$, whence tang. $b=\frac{1}{3}$.

Hence $45°=$the arc whose tangent is $\frac{1}{2}+$the arc whose tangent is $\frac{1}{3}$.

Substituting $\frac{1}{2}$ and $\frac{1}{3}$ for x in equation (1), Art. 57, we have

$$45°=\frac{\pi}{4}=\frac{1}{2}-\frac{1}{3.2^3}+\frac{1}{5.2^5}-\frac{1}{7.2^7}+\text{ etc.}$$

$$+\frac{1}{3}-\frac{1}{3.3^3}+\frac{1}{5.3^5}-\frac{1}{7.3^7}+\text{ etc.}$$

In order to compute the value of this, or any similar series, reduce each term to a decimal fraction. A few terms of this series give

$$\frac{\pi}{4}=0.4636476+0.3217506=0.785398.$$

Hence

$$\pi=3.141592+.$$

59. *When Maclaurin's theorem fails.* Maclaurin's theorem enables us to develop any function of one variable, whenever the function can be developed in the proposed form. But there are functions which can not be expanded in a series of positive integral powers of the variable. In such a case the function, or some one of its successive differential coefficients, becomes ∞ when the variable is made equal to 0.

For example, let $u=x^{\frac{1}{2}}$. Hence $(u)=0$.

$$\frac{du}{dx}=\frac{1}{2x^{\frac{1}{2}}}. \quad \text{“} \quad \left(\frac{du}{dx}\right)=\frac{1}{0}=\infty.$$

$$\frac{d^2u}{dx^2}=-\frac{1}{4x^{\frac{3}{2}}}. \quad \text{“} \quad \left(\frac{d^2u}{dx^2}\right)=-\infty.$$

Substituting these values in Maclaurin's theorem, we have

$$u=x^{\frac{1}{2}}=0+\infty x-\infty.\frac{x^2}{2}+\text{ etc.},$$

a series of which we can make no use. In this example, Maclaurin's theorem is said to *fail*, and it fails because the function $x^{\frac{1}{2}}$ can not be expanded in the particular form assumed in the theorem.

So also the functions $u=\frac{1}{x}$, $u=\cot.\ x$, $u=\log.\ x$, can not be developed by Maclaurin's theorem in integral powers of x. Such functions can, however, often be developed in fractional powers of x, or in powers of some simple function of x; as, for instance, in powers of $1+x$.

60. *Function of the sum of two variables.* A quantity is said to be a function of the sum of two variables when in the algebraic expression for the function a single variable may be substituted for the sum, and the original function be thus reduced to a function of the single variable of the same form as the primitive function.

Thus, let $u=a(x+y)^n$.

If in place of $x+y$ we substitute z, the function becomes $u=az^n$, which is a function of z, of the same form as $a(x+y)^n$.

So also the function $u=\log.(x-y)$ becomes $u=\log.\ z$ when we put z for $x-y$. Here u is a function of the sum of x and $-y$.

61. *Partial differential coefficients.* Let u be a function of two independent variables x and y; then, since the variables are independent, however either be supposed to vary, the

other will remain unchanged, the function ought to furnish two differential coefficients; the one resulting from ascribing a variation to x, and the other from ascribing a variation to y; y entering the first coefficient as if it were a constant, and x entering the second as if it were a constant.

If we suppose y to remain constant and x to vary, the differential coefficient will be $\frac{du}{dx}$; and if we suppose x to remain constant and y to vary, the differential coefficient will be $\frac{du}{dy}$.

The differential coefficients which are obtained under these suppositions are called *partial differential coefficients.* The first is the partial differential coefficient taken with respect to x, and the second is the partial differential coefficient taken with respect to y.

TAYLOR'S THEOREM.

62. Taylor's theorem is a theorem for expanding a function of the sum of two variables into a series arranged according to the ascending powers of one of the variables, with coefficients that are functions of the other variable, and depend also upon the constants which enter the given function.

The demonstration of Taylor's theorem is founded upon the principle that if $u=f(x+y)$ be any function of $x+y$, the partial differential coefficient of u will be the same, whether we suppose x to remain constant and y to vary, or y to remain constant and x to vary; that is, the partial differential coefficients $\frac{du}{dx}$ and $\frac{du}{dy}$ are equal.

Assume $z=x+y$, then $u=f(z)$.

If we differentiate this as a function of x, y being regarded as a constant, we shall have, Art. 33,

$$\frac{du}{dx}=\frac{du}{dz}\times\frac{dz}{dx}.$$

If we differentiate it as a function of y, x being regarded as a constant, we shall have

$$\frac{du}{dy}=\frac{du}{dz}\times\frac{dz}{dy}.$$

But, since $z=x+y$, when y is regarded as constant, $dz=dx$; that is,

$$\frac{dz}{dx}=1.$$

Also, when x is regarded as constant, $dz=dy$; that is, $\frac{dz}{dy}=1$.

Hence $$\frac{du}{dx}=\frac{du}{dz}, \text{ and } \frac{du}{dy}=\frac{du}{dz};$$

or $$\frac{du}{dx}=\frac{du}{dy};$$

that is, *the partial differential coefficient of the sum of two variables, x and y, obtained on the supposition that x is variable and y constant, is the same as that obtained on the supposition that y is variable and x constant.*

Ex. 1. Find the two partial differential coefficients of $u=(x+y)^n$. *Ans.* $\frac{du}{dx}=n(x+y)^{n-1}$, and $\frac{du}{dy}=n(x+y)^{n-1}$.

Ex. 2. Find the two partial differential coefficients of $u=(x+y)^{\frac{1}{3}}$. *Ans.* $\frac{du}{dx}=\frac{1}{3}(x+y)^{-\frac{2}{3}}$, and $\frac{du}{dy}=\frac{1}{3}(x+y)^{-\frac{2}{3}}$.

Ex. 3. Find the two partial differential coefficients of $u=\log.(x+y)$. *Ans.* $\frac{du}{dx}=\frac{1}{x+y}$, and $\frac{du}{dy}=\frac{1}{x+y}$.

63. *Product or quotient of two variable quantities.* The preceding principle will *not* hold true for the partial differential coefficients of the product or quotient of two variable quantities.

Ex. 1. Find the two partial differential coefficients of $u=\left(\frac{x}{y}\right)^n$. *Ans.* $\frac{du}{dx}=\frac{nx^{n-1}}{y^n}$, and $\frac{du}{dy}=\frac{-nx^n}{y^{n+1}}$.

Ex. 2. Find the two partial differential coefficients of $u=\log.(xy)$. *Ans.* $\frac{du}{dx}=\frac{1}{x}$, and $\frac{du}{dy}=\frac{1}{y}$.

64. *Demonstration of Taylor's theorem.* Let u' be any function of the sum of two variables x and y. It is required to determine the law of the development when the function can be expanded in the form

$$u' = f(x+y) = A + By + Cy^2 + Dy^3 + \text{ etc.}, \quad (1)$$

where A, B, C, D, etc., are independent of y, but dependent upon x, and upon the constants which enter the primitive function. It is required to find such values for A, B, C, etc., as shall render the development true for all possible values of x and y.

If we differentiate upon the supposition that x varies and y remains constant, and divide the result by dx, we shall have

$$\frac{du'}{dx} = \frac{dA}{dx} + \frac{dB}{dx}y + \frac{dC}{dx}y^2 + \frac{dD}{dx}y^3 + \text{ etc.}$$

If we differentiate upon the supposition that y varies and x remains constant, and divide by dy, we shall have

$$\frac{du'}{dy} = B + 2Cy + 3Dy^2 + \text{ etc.}$$

But, by Art. 62, we have $\frac{du'}{dx} = \frac{du'}{dy}$; therefore

$$\frac{dA}{dx} + \frac{dB}{dx}y + \frac{dC}{dx}y^2 + \text{ etc.} = B + 2Cy + 3Dy^2 + \text{ etc.}$$

Now since this equality exists whatever be the value of y, the coefficients of the like powers of y in the two members must be equal to each other (Algebra, Art. 378); therefore,

$$\frac{dA}{dx} = B, \quad (2)$$

$$\frac{dB}{dx} = 2C, \quad (3)$$

$$\frac{dC}{dx} = 3D, \text{ etc.} \quad (4)$$

But, since equation (1) is true for all values of y, we may make $y=0$, in which case function of $x+y$ will reduce to function of x, which we will denote by u. Therefore, $A = u$.

Substituting this value of A in equation (2), we have

$$B=\frac{du}{dx}.$$

Substituting this value of B in equation (3), we have

$$2C=\frac{d\,.\,du}{dx\,.\,dx}=\frac{d^2u}{dx^2};\text{ whence }C=\frac{d^2u}{2dx^2}.$$

Substituting this value of C in equation (4), we have

$$3D=\frac{d\,.\,d^2u}{2dx\,.\,dx^2}=\frac{d^3u}{2dx^3};\text{ whence }D=\frac{d^3u}{2\,.\,3dx^3},\text{ and so on.}$$

Substituting these values of A, B, C, D, etc., in equation (1), we have

$$u'=f(x+y)=u+\frac{du}{dx}\cdot\frac{y}{1}+\frac{d^2u}{dx^2}\cdot\frac{y^2}{1.2}+\frac{d^3u}{dx^3}\cdot\frac{y^3}{1.2.3}+\text{ etc.},$$

which is known by the name of *Taylor's theorem.*

It will thus be seen that Taylor's theorem develops a function of the sum of x and y into a series of terms that contain increasing powers of y multiplied by coefficients which are functions of x. The first term is the value of the original function when $y=0$; the second term is y multiplied by the first differential coefficient of the first term with respect to x; the third term is $\frac{y^2}{2}$ multiplied by the second differential coefficient of the first term with respect to x; etc.

The development of $f(x-y)$ is obtained from the same theorem by changing $+y$ into $-y$, thus,

$$u'=f(x-y)=u-\frac{du}{dx}\cdot\frac{y}{1}+\frac{d^2u}{dx^2}\cdot\frac{y^2}{1.2}-\frac{d^3u}{dx^3}\cdot\frac{y^3}{1.2.3}+\text{ etc.}$$

Ex. 1. Expand $u'=(x+y)^n$ by Taylor's theorem.

Making $y=0$, we obtain $u=x^n$, and thence, by differentiation,

$$\frac{du}{dx}=nx^{n-1},\quad\frac{d^2u}{dx^2}=n(n-1)x^{n-2},$$

$$\frac{d^3u}{dx^3}=n(n-1)(n-2)x^{n-3},\text{ etc.}$$

These values being substituted in the formula, give

$$u'=(x+y)^n=x^n+nx^{n-1}y+\frac{n(n-1)}{1.2}x^{n-2}y^2+\frac{n(n-1)(n-2)}{1.2.3}x^{n-3}y^3+\text{ etc.},$$

the same as found by the binomial theorem.

Ex. 2. Expand $u'=(x-y)^{-4}$ by Taylor's theorem.

$$Ans.\ (x-y)^{-4}=x^{-4}+4x^{-5}y+10x^{-6}y^2+20x^{-7}y^3+35x^{-8}y^4+\text{etc.}$$

Ex. 3. Expand $u'=(x-y)^{\frac{1}{3}}$ by Taylor's theorem.

$$Ans.\ (x-y)^{\frac{1}{3}}=x^{\frac{1}{3}}-\tfrac{1}{3}x^{-\frac{2}{3}}y-\frac{2}{3.6}x^{-\frac{5}{3}}y^2-\frac{2.5}{3.6.9}x^{-\frac{8}{3}}y^3-\text{etc.}$$

Ex. 4. Expand $u'=\sin.(x+y)$ by Taylor's theorem.

$$Ans.\ \sin.(x+y)=\sin.x\left\{1-\frac{y^2}{1.2}+\frac{y^4}{1.2.3.4}-\frac{y^6}{1.2.3.4.5.6}+\text{etc.}\right\}$$

$$+\cos.x\left\{y-\frac{y^3}{1.2.3}+\frac{y^5}{1.2.3.4.5}-\frac{y^7}{1.2.3.4.5.6.7}+\text{etc.}\right\}$$

Ex. 5. Expand $u'=\cos.(x+y)$ by Taylor's theorem.

$$Ans.\ \cos.(x+y)=\cos.x\left\{1-\frac{y^2}{1.2}+\frac{y^4}{1.2.3.4}-\frac{y^6}{1.2.3.4.5.6}+\text{etc.}\right\}$$

$$-\sin.x\left\{y-\frac{y^3}{1.2.3}+\frac{y^5}{1.2.3.4.5}-\frac{y^7}{1.2.3.4.5.6.7}+\text{etc.}\right\}$$

Ex. 6. Expand $u'=\log.(x+y)$ by Taylor's theorem.

$$Ans.\ \log.(x+y)=\log.x+\frac{y}{x}-\frac{y^2}{2x^2}+\frac{y^3}{3x^3}-\frac{y^4}{4x^4}+\text{etc.}$$

Ex. 7. Expand $u'=a^{x+y}$ by Taylor's theorem.

$$Ans.\ a^{x+y}=a^x\left\{1+y.\log.a+\frac{y^2}{1.2}(\log.a)^2+\frac{y^3}{1.2.3}(\log.a)^3+\text{etc.}\right\}$$

65. *When Taylor's theorem fails.* Taylor's theorem will give the expansion of $(x+y)$ in all cases so long as x retains its *general* value; but in certain functions a particular value may be given to x which will render some of the differential coefficients infinite; in such cases the theorem is said to *fail* in giving the development. In the demonstration of the theorem, Art. 64, when the coefficients of the like powers of y in the two members of the equation were assumed to be equal, these coefficients were supposed to be finite quantities. Hence, if such a value should be assigned to one of the variables as to make any one of these coefficients indefinitely great, the demonstration would be inconclusive. In such a case the theorem fails in giving the development; and this failure indi-

cates that the particular value of the function proposed can not be expressed in integral positive powers of y combined with *finite* coefficients.

Ex. 1. Develop $u'=(x+y+a)^{\frac{1}{2}}$ by Taylor's theorem.

Here
$$u=(x+a)^{\frac{1}{2}}$$
$$\frac{du}{dx}=\tfrac{1}{2}(x+a)^{-\frac{1}{2}}=\frac{1}{2(x+a)^{\frac{1}{2}}};$$
$$\frac{d^2u}{dx^2}=-\tfrac{1}{4}(x+a)^{-\frac{3}{2}}=-\frac{1}{4(x+a)^{\frac{3}{2}}};$$
$$\frac{d^3u}{dx^3}=\tfrac{3}{8}(x+a)^{-\frac{5}{2}}=\frac{3}{8(x+a)^{\frac{5}{2}}};\text{ etc.}$$

Substituting in Taylor's theorem, we have
$$u'=(x+y+a)^{\frac{1}{2}}=(x+a)^{\frac{1}{2}}+\frac{y}{2(x+a)^{\frac{1}{2}}}-\frac{y^2}{8(x+a)^{\frac{3}{2}}}+\frac{3y^3}{48(x+a)^{\frac{5}{2}}}-\text{etc.},$$
which is the *general* development of u'. But if we suppose $x=-a$, the series becomes
$$u'=y^{\frac{1}{2}}=0+\infty-\infty+\infty-\text{ etc.},$$
from which nothing can be determined except that the square root of y can not be exhibited in integral positive powers of y with finite coefficients.

Ex. 2. For what value of x does the development of $u'=(x+y+b)^{\frac{1}{n}}$ by Taylor's theorem fail? *Ans.* $x=-b$.

DIFFERENTIATION OF FUNCTIONS OF TWO INDEPENDENT VARIABLES.

66. Let u be a function of x and y, where each variable is supposed to be entirely independent of the other. We will suppose the variables to change separately, x first receiving an increment h, and y afterward receiving an increment k.

When x receives the increment h, $f(x,y)$ becomes $f(x+h,y)$; and if y be regarded as constant, this function may be developed by Taylor's theorem; whence
$$u'=f(x+h,y)=u+\frac{du}{dx}h+\frac{d^2u}{dx^2}\frac{h^2}{2}+\text{ etc.},\qquad(1)$$

in which $\frac{du}{dx}$, $\frac{d^2u}{dx^2}$, etc., are the differential coefficients of u taken under the supposition that x alone is variable, and they are all functions of x and y.

If in equation (1) we suppose y to become $y+k$, each term will receive a change. The first member becomes $f(x+h, y+k)$. The first term u of the second member may be expanded by Taylor's theorem, and becomes

$$u+\frac{du}{dy}k+\frac{d^2u}{dy^3}\frac{k^2}{2}+\text{ etc.}$$

In like manner $\frac{du}{dx}$ in the second term of equation (1) may be expanded, and becomes

$$\frac{du}{dx}+\frac{d.\frac{du}{dx}}{dy}k+\text{ etc.},$$

or

$$\frac{du}{dx}+\frac{d^2u}{dxdy}k+\text{ etc.}$$

So also $\frac{d^2u}{dx^2}$ in the third term may be expanded into a series of terms, of which $\frac{d^2u}{dx^2}$ is the first, and is the only one which does not contain k as a factor.

Making these substitutions in equation (1), we have

$$\begin{aligned} u''=f(x+h, y+k)=u+\frac{du}{dy}k+\frac{d^2u}{dy^2}\frac{k^2}{2}+\text{ etc.},\\ +\frac{du}{dx}h+\frac{d^2u}{dx.dy}hk+\text{ etc.},\\ +\frac{d^2u}{dx^2}\frac{h^2}{2}+\text{ etc.}, \qquad (2) \end{aligned}$$

where each of the omitted terms involves more than two dimensions of h and k.

If from each member of equation (2) we subtract $f(x, y)$, we have $u''-u=\frac{du}{dx}h+\frac{du}{dy}k+$ other terms involving at least two dimensions of h and k. This is the expression for the increment which the function u has received from attribu-

ting an increment to each of the variables x and y; that is,

$$\text{incr. } u = \frac{du}{dx}\,\text{incr. } x + \frac{du}{dy}\,\text{incr. } y + \text{terms involving two dimensions of } h \text{ and } k.$$

Dividing each member by incr. u, we have

$$1 = \frac{du\,.\,\text{incr. } x}{dx\,.\,\text{incr. } u} + \frac{du\,.\,\text{incr. } y}{dy\,.\,\text{incr. } u} + \text{X},$$

where X represents the omitted terms, each of which contains as a factor h or k in at least two dimensions, and also contains *increment* u as a divisor.

Now, to determine the value of these ratios at their limit, we must make h and k each equal to zero, in which case the term X disappears, as in Art. 18, and we have

$$1 = \frac{dudx}{dx\mathrm{D}u} + \frac{dudy}{dy\mathrm{D}u},$$

or

$$\mathrm{D}u = \frac{dudx}{dx} + \frac{dudy}{dy};$$

where Du in the first member denotes what is called the *total differential* of the function, while the first du in the second member denotes what is called the *partial differential* taken with respect to x, and the second du denotes the *partial differential* taken with respect to y.

The total differential Du is by some authors written with the small letter, as du, in which case the partial differential is made the numerator of a fraction, as $\frac{du}{dx}$ or $\frac{du}{dy}$, which are partial differential coefficients of u with respect to x and to y.

67. *Total differential.* The differential of a function of two independent variables, which is obtained upon the supposition that both the variables have changed their values, is called the total differential of the function; and we see from Art. 66 that *the total differential of a function of two independent variables is equal to the sum of the partial differentials.*

In the same manner it may be proved that *the total differential of a function of any number of variables is the sum of the several partial differentials arising from differentiating the function relatively to each variable in succession as if all the others were constants.*

Ex. 1. If one side of a rectangle increase at the rate of 1 inch per second, and the other at the rate of 2 inches, at what rate is the area increasing when the first side becomes 8 inches, and the last 12 inches? *Ans.* 28 inches per second.

Ex. 2. If one side of a rectangle increase at the rate of 2 inches per second, and the other diminish at the rate of 3 inches per second, at what rate is the area increasing or diminishing when the first side becomes 10 inches, and the second 8?

Ans.

Ex. 3. If the major axis of an ellipse increase uniformly at the rate of 2 inches per second, and the minor axis at the rate of 3 inches, at what rate is the area increasing when the major axis becomes 20 inches, the minor axis at the same instant being 12 inches? *Ans.* 21π inches.

Ex. 4. If the altitude of a cone diminish at the rate of 3 inches per second, and the diameter of the base increase at the rate of 1 inch per second, at what rate does the solidity vary when the altitude becomes 18 inches, the diameter of the base at the same instant being 10 inches? *Ans.*

Ex. 5. Find the total differential of $u=x^3+ax^2y+y^3$.

$$\frac{dudx}{dx}=(3x^2+2axy)dx,$$

$$\frac{dudy}{dy}=(ax^2+3y^2)dy.$$

Hence $$Du=(3x^2+2axy)dx+(ax^2+3y^2)dy.$$

Ex. 6. Find the total differential of $u=\frac{x+y}{x-y}$.

Ans. $Du=\frac{2xdy-2ydx}{(x-y)^2}$.

Ex. 7. Find the total differential of $u=\frac{x}{y}+\frac{y}{x}$.

$$Ans.\ Du=\frac{dx}{y}-\frac{ydx}{x^2}+\frac{dy}{x}-\frac{xdy}{y^2}.$$

Ex. 8. Find the total differential of $u=ax^2y^3z^4$.

$$Ans.\ Du=2axy^3z^4dx+3ax^2y^2z^4dy+4ax^2y^3z^3dz.$$

Ex. 9. Find the total differential of $u=x^y$.

$$Ans.\ Du=yx^{y-1}dx+x^y\log.xdy.$$

Ex. 10. Find the total differential of $u=\text{tang.}^{-1}\frac{x}{y}$.

$$Ans.\ Du=\frac{ydx-xdy}{y^2+x^2}.$$

Ex. 11. Find the total differential of $u=\sin.(xy)$.

$$Ans.\ Du=\cos.(xy)\{ydx+xdy\}.$$

DIFFERENTIAL EQUATIONS.

68. If we differentiate the equation of a line, we shall obtain a new equation, which expresses the relation between the differentials of the co-ordinates of the line. This equation is called the *differential equation of the line.*

If we take the equation of a straight line

$$y=mx+c, \tag{1}$$

and differentiate it, we obtain

$$\frac{dy}{dx}=m, \tag{2}$$

a result that is the same for all values of c, and is equally applicable to every line that is parallel to $y=mx$.

Differentiating equation (2), we obtain

$$\frac{d^2y}{dx^2}=0.$$

This last equation is entirely independent of the values of the constants m and c, and is equally applicable to every straight line which can be drawn in the plane of the co-ordinate axes. It is called the *general differential equation of lines of the first order.*

69. If we take the equation of the circle

$$y^2=r^2-x^2, \tag{1}$$

and differentiate it, we obtain

$$2ydy=-2xdx;$$

whence

$$\frac{dy}{dx}=-\frac{x}{y}. \tag{2}$$

Equation (2) is independent of the value of the radius r, and hence it belongs equally to every circle having its origin at the centre.

If we take the equation of the parabola

$$y^2=4ax, \tag{1}$$

and differentiate it, we obtain

$$2ydy=4adx;$$

whence

$$\frac{dy}{dx}=\frac{2a}{y}. \tag{2}$$

But from equation (1) we have

$$2a=\frac{y^2}{2x};$$

hence equation (2) becomes

$$\frac{dy}{dx}=\frac{y}{2x}.$$

This equation is independent of the value of the latus rectum $4a$, and hence it belongs equally to every parabola whose principal vertex is at the origin, and whose axis is the axis of x.

70. If we take the general equation of lines of the second order, which is (Anal. Geom., Art. 234)

$$y^2=mx+nx^2, \tag{1}$$

and differentiate it, we obtain

$$2ydy=mdx+2nxdx. \tag{2}$$

Differentiating again, regarding dx as constant, we obtain

$$2dy^2+2yd^2y=2ndx^2,$$

or, dividing by 2,

$$dy^2+yd^2y=ndx^2. \tag{3}$$

By combining equations (1), (2), and (3), we may eliminate the constants m and n, and we obtain

$$y^2dx^2+x^2dy^2+yx^2d^2y-2xydxdy=0,$$

which is the *general differential equation of lines of the second order*, and is applicable to every conic section of which the axis of x is a diameter and the origin is a principal vertex.

71. Thus we see that an equation may be freed from its constants by successive differentiations; and for this purpose it is necessary to differentiate it as many times as there are constants to be eliminated. The differential equations thus obtained, together with the given equation, make one equation more than the number of constants to be eliminated, and hence we may derive from them a new equation, which will be freed from these constants.

The equation which is thus obtained is the general differential equation of the *species of lines* one of which is represented by the given equation, since, being independent of the constants, it belongs to all lines of the same species referred to the same co-ordinate axes.

Generally it is not difficult, by differentiation and elimination of constants, to deduce differential equations; but to pass from differential equations to the primitive equation from which they were deduced is more difficult, and forms one of the most important branches of the mathematics. In fact, nearly all investigations in physical science lead to differential equations, from which it is required to discover the relation subsisting among the primitive variables.

VANISHING FRACTIONS.

72. It sometimes happens that the substitution of a particular value for the variable in a fraction causes both the numerator and denominator to vanish, and the fraction reduces to $\frac{0}{0}$. Such fractions are called *vanishing fractions*. Thus $\frac{x-a}{x^2-a^2}$ becomes $\frac{0}{0}$ when $x=a$.

This form arises from the existence of a factor common to

both numerator and denominator, which factor becomes 0 under the particular supposition. Hence, if we can by any means eliminate this factor before substituting a for x, we shall be able to find the true value of the fraction.

Sometimes the vanishing factor is readily seen, and may be immediately canceled; as in the above fraction, which contains the factor $x-a$ in both numerator and denominator. Canceling this factor, the fraction becomes $\frac{1}{x+a}$, whose value is $\frac{1}{2a}$ when $x=a$. This is, therefore, the true value of the fraction $\frac{x-a}{x^2-a^2}$ when $x=a$.

So also the fraction

$$\frac{x^2-ax}{bx-ab}=\frac{0}{0} \text{ when } x=a;$$

but rejecting the common factor $x-a$, the fraction becomes

$$\frac{x}{b}, \text{ which reduces to } \frac{a}{b} \text{ when } x=a.$$

In many other cases the true value of a vanishing fraction may be ascertained by performing a few simple transformations. Thus

$$\frac{(x^2-a^2)^{\frac{1}{2}}+(x-a)}{(x-a)^{\frac{1}{2}}+(x^2-a^2)^{\frac{1}{2}}}=\frac{0}{0} \text{ when } x=a;$$

but dividing numerator and denominator by the common factor $(x-a)^{\frac{1}{2}}$, we have

$$\frac{(x+a)^{\frac{1}{2}}+(x-a)^{\frac{1}{2}}}{1+(x+a)^{\frac{1}{2}}}=\frac{\sqrt{2a}}{1+\sqrt{2a}} \text{ when } x=a.$$

73. The differential calculus affords a more direct and general method of finding the value of a vanishing fraction. Let the fraction be $\frac{N}{D}$; then, if N and D both vanish when a particular value a is assigned to the variable, they must have a common divisor consisting of some power of $x-a$.

First suppose $\frac{N}{D}=\frac{P(x-a)}{Q(x-a)}$, in which P and Q are functions of x, but do not contain the factor $x-a$. Differentiating, we have

$$\frac{dN}{dD}=\frac{dP(x-a)+Pdx}{dQ(x-a)+Qdx},$$

which becomes $\frac{P}{Q}$ when we substitute a for x, and since P and Q do not vanish when we substitute a for x, $\frac{P}{Q}$ is the required value of the fraction.

Next suppose the common factor to be $(x-a)^2$, or

$$\frac{N}{D}=\frac{P(x-a)^2}{Q(x-a)^2};$$

then

$$\frac{dN}{dD}=\frac{dP(x-a)^2+2P(x-a)dx}{dQ(x-a)^2+2Q(x-a)dx},$$

which $=\frac{0}{0}$ when a is substituted for x.

Differentiating a second time, we have

$$\frac{d^2N}{d^2D}=\frac{d^2P(x-a)^2+4dP(x-a)dx+2Pdx^2}{d^2Q(x-a)^2+4dQ(x-a)dx+2Qdx^2},$$

which becomes $\frac{P}{Q}$ when we substitute a for x, and since P and Q do not vanish when $x=a$, $\frac{P}{Q}$ is the required value of the fraction.

Generally, if $\frac{N}{D}=\frac{P(x-a)^m}{Q(x-a)^m}$, where m is a positive integer, by differentiating N and D m times, and substituting a for x in the result, we shall obtain $\frac{N}{D}=\frac{P}{Q}$.

74. If $\frac{N}{D}=\frac{P(x-a)^m}{Q(x-a)^n}=\frac{P}{Q}(x-a)^{m-n}$, and we differentiate n times, and substitute a for x in the result, we shall have $\frac{N}{D}=\frac{0}{Q}=0$, m being greater than n; but if m be less than n,

differentiating m times we shall have $\frac{N}{D}=\infty$. Hence, to find the value of a vanishing fraction, *for the numerator and denominator substitute their first differential coefficients, their second differential coefficients, and so on, till we obtain a fraction in which numerator and denominator do not both vanish for $x=a$; the value thus found will be the true value of the vanishing fraction.* If the numerator vanish when the denominator is finite, the fraction $=0$, but if the denominator vanish when the numerator is finite, the fraction $=\infty$.

The preceding demonstration is only applicable to algebraic functions; but it may also be shown of transcendental functions, that when they take the form of $\frac{0}{0}$ they have a determinate value, which may be found by successive differentiations.

Ex. 1. Find the value of $u=\frac{x^3-1}{x^3+2x^2-x-2}$ when $x=1$.

Here the differential coefficient of the numerator is $3x^2$, and that of the denominator is $3x^2+4x-1$.

$$\therefore u=\frac{3x^2}{3x^2+4x-1}=\tfrac{1}{2} \text{ when } x=1.$$

Ex. 2. Find the value of $u=\frac{x^n-1}{x-1}$ when $x=1$.

Ans. $u=n$.

Ex. 3. Find the value of $u=\frac{x^4-a^4}{x^2-a^2}$ when $x=a$.

Ans. $u=2a^2$.

Ex. 4. Find the value of $u=\frac{x^3-x^2-x+1}{x^4-2x^3+x^2}$ when $x=1$.

Ans. $u=2$, from two differentiations.

Ex. 5. Find the value of $u=\frac{3x^2-6x+3}{2x^2-4x+2}$ when $x=1$.

Ans. $u=\frac{3}{2}$.

Ex. 6. Find the value of $u=\frac{ax^2-2acx+ac^2}{bx^2-2bcx+bc^2}$ when $x=c$.

Ans. $u=\frac{a}{b}$.

Ex. 7. Find the value of $u=\frac{a^x-b^x}{x}$ when $x=0$.

Ans. $u=\log.\left(\frac{a}{b}\right)$.

Ex. 8. Find the value of $u=\frac{x^2-ax}{x^4-2ax^3+2a^3x-a^4}$ when $x=a$.

Ans. $u=\infty$.

Ex. 9. Find the value of $u=\frac{1-\sin.x}{\cos.x}$ when $x=90°$.

Ans. $u=0$.

75. The method of Art. 74 is not applicable when the common factor $x-a$ of the numerator and denominator is a radical. For, since at each differentiation the exponent m of the factor is diminished by unity, if the exponent is a proper fraction it will become negative without reducing to zero, and the fraction will take the form of $\frac{\infty}{\infty}$.

In such a case, we may obtain the true value of the fraction by substituting $a+h$ for x both in the numerator and denominator of the fraction, reducing the result to its simplest form, and then making the increment equal to zero. It is plain that this will amount to the same as the substitution of a for x; but by the reduction of the fraction to its lowest terms, the common factor is made to disappear, and we obtain the true value of the fraction for the particular value of the variable.

The substitution of a particular value for the variable may cause a function to assume the form of $0\times\infty$, 0^0, or ∞^0, etc. Such expressions can sometimes be reduced to the form $\frac{0}{0}$, and their values be determined by the methods already explained.

Ex. 1. Find the value of $u=\frac{(x^2-a^2)^{\frac{3}{2}}}{(x-a)^{\frac{3}{2}}}$ when $x=a$.

Assume $x=a+h$;

then $\frac{N}{D}=\frac{(2ah+h^2)^{\frac{3}{2}}}{h^{\frac{3}{2}}}=(2a+h)^{\frac{3}{2}}$,

which, when $h=0$, becomes $(2a)^{\frac{3}{2}}$, which is the value of the given fraction when $x=a$.

Ex. 2. Find the value of $u=\dfrac{(x^2-3ax+2a^2)^{\frac{2}{3}}}{(x^3-a^3)^{\frac{1}{2}}}$ when $x=a$.

Assume $x=a+h$;

then

$$\frac{N}{D}=\frac{(h^2-ah)^{\frac{2}{3}}}{(3a^2h+3ah^2+h^3)^{\frac{1}{2}}}=\frac{h^{\frac{1}{6}}(h-a)^{\frac{2}{3}}}{(3a^2+3ah+h^2)^{\frac{1}{2}}}=\frac{0}{(3a^2)^{\frac{1}{2}}} \text{ when } x=a,$$

or $$u=\frac{N}{D}=0.$$

Ex. 3. Find the value of $u=\dfrac{x^{\frac{1}{2}}-a^{\frac{1}{2}}+(x-a)^{\frac{1}{2}}}{(x^2-a^2)^{\frac{1}{2}}}$ when $x=a$.

Ans. $u=\dfrac{1}{(2a)^{\frac{1}{2}}}$.

Ex. 4. Find the value of $u=\dfrac{\sqrt{2a^3x-x^4}-\sqrt{ax^3}}{a-\sqrt{ax}}$ when $x=a$.

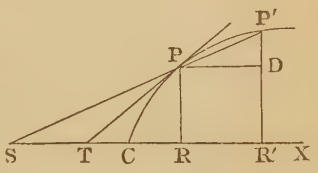
Ans. $u=5a$.

CHAPTER IV.

GEOMETRICAL REPRESENTATION OF THE FIRST DIFFERENTIAL COEFFICIENT. MAXIMA AND MINIMA VALUES OF A FUNCTION.

76. *Geometrical representation of a function.* Every function of a single variable may be represented by the ordinate of a curve, of which the variable is the corresponding abscissa. For let y be an explicit function of a variable x expressed by an equation $y=f(x)$,
and let any value be assigned to x, and the corresponding value of y be deduced. Suppose x and y to be the co-ordinates of a point referred to two rectangular axes; then for each value of x there will be one or more values of y, and these, if real, will determine the position of points with respect to the given axes. The points thus determined will lie upon and represent a locus or curve, and in general the relation between the ordinate and abscissa of any point of the curve will be the same as that between the given function and the independent variable.

77. *First differential coefficient.* Let CPP′ be a curve whose equation is $y=f(x)$, and let P be any point of this curve whose co-ordinates are x and y. Increase the abscissa CR or x by the arbitrary increment RR′, which we will represent by h. Denote the corresponding ordinate P′R′ by y'; draw the secant SPP′, the tangent PT, and draw PD parallel to CX. Then



$$P'D=P'R'-PR=y'-y=Ah+Bh^2 \text{ (Art. 23).}$$

But from the triangle P′DP we have

$$PD : P'D :: 1 : \text{tang. } P'PD=\frac{P'D}{PD},$$

and substituting for P'D and PD their values, we have

$$\text{tang. P'PD} = \text{tang. PSR} = \frac{Ah + Bh^2}{h} = A + Bh,$$

which expresses the ratio of the increment of y to that of x. In order to find the differential coefficient of y with respect to x, we must determine the limit of this ratio by making the increment equal to zero (Art. 18).

Now, when h is diminished, the point P' approaches P, the secant PS approaches the tangent PT; and finally, since the limiting position of the secant is the tangent, we have at the limit, making $h = 0$,

$$\text{tang. PTX} = A = \frac{dy}{dx};$$

that is, *the tangent of the angle which a tangent line at any point of a curve makes with the axis of abscissas is equal to the first differential coefficient of the ordinate of the curve.*

78. *Given values of the first differential coefficient.* If it is required to find the point of a given curve at which the tangent line makes a given angle with the axis of X, we know that at this point the first differential coefficient must be equal to the tangent of the given angle. If we represent this tangent by a, we must have

$$\frac{dy}{dx} = a,$$

and this, combined with the equation of the curve, will give the particular values of x and y for the required point.

If the tangent line is to be parallel to the axis of X, the angle which it forms with that axis is zero, and its tangent is zero; hence, for the point of tangency $\frac{dy}{dx} = 0$.

If the tangent line is to be perpendicular to the axis of X, then, since the tangent of $90° = \infty$, $\frac{dy}{dx} = \infty$. That is, *for a point at which the tangent to a curve is parallel to the axis of abscissas, the first differential coefficient is equal to zero; and*

for a point at which the tangent to a curve is perpendicular to the axis of abscissas, the first differential coefficient is equal to infinity.

Ex. 1. It is required to find the point on a given parabola at which the tangent line makes an angle of 45° with the axis.

The equation of the parabola (Anal. Geom., Art. 85) is

$$y^2=4ax.$$

Differentiating, we obtain

$$2ydy=4adx,$$

or

$$\frac{dy}{dx}=\frac{2a}{y}.$$

But, since tang. 45° equals radius or unity, we have

$$\frac{2a}{y}=1, \text{ or } 2a=y.$$

Combining this with the equation $y^2=4ax$, we find

$$x=a.$$

Hence the required tangent passes through the extremity of the ordinate drawn from the focus.

Ex. 2. It is required to determine at what point the tangent to the circumference of a circle is parallel to the axis of X.

The equation of the circle (Anal. Geom., Art. 60) is

$$x^2+y^2=r^2.$$

By differentiating we obtain

$$\frac{dy}{dx}=-\frac{x}{y},$$

and placing this equal to 0 we find

$$x=0.$$

But when $x=0$ we have $y=\pm r$;

hence the tangent is parallel to the axis of X at the two points where the circumference intersects the axis of ordinates.

Ex. 3. It is required to determine at what point the tangent to an ellipse is perpendicular to the axis of X.

The equation of the ellipse (Anal. Geom., Art. 121) is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

By differentiating we obtain

$$\frac{dy}{dx} = -\frac{b^2x}{a^2y},$$

and placing this equal to ∞, we have

$$-\frac{b^2x}{a^2y} = \infty,$$

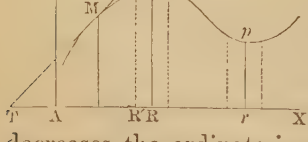
whence $y=0$, and $x=\pm a$.

Hence the tangent is perpendicular to the axis of X at either extremity of the transverse axis.

MAXIMA AND MINIMA OF FUNCTIONS OF A SINGLE VARIABLE.

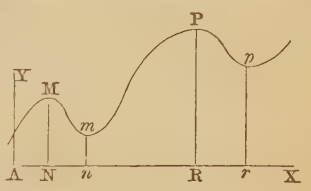
79. *Definition of maximum and minimum.* Suppose u to be a certain function of x, and that while the variable x changes gradually from one definite value to another, u changes in such a manner that it is sometimes increasing and sometimes decreasing. There must, then, be a certain value of x, for which u, having previously been increasing, begins to decrease; or begins to increase, having previously been decreasing. In the former case u has a greater value for this particular value of x than it has for adjacent values of x, and it is said to have a *maximum* value. In the latter case u has a less value for the particular value of x than it has for adjacent values of x, and is said to have a *minimum* value. We say, then, that a function of a single variable is at a maximum value, or is *a maximum, when it is greater than any of the values which immediately precede or follow it; and is at a minimum value when it is less than any of the values which immediately precede or follow it.*

Thus, if we suppose the function u to be represented by the variable ordinate P′R′ of a curve line, and if the ordinate moving from A along AX gradually increases until it comes into the position PR, and after that gradually decreases, the ordinate is said to be a maximum, or at a maximum value, when it comes into the position PR.

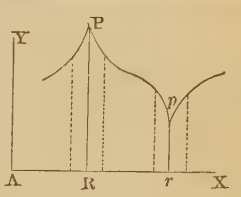


But if the ordinate moving along AX gradually decreases until it comes into the position pr, and after that gradually increases, the ordinate is said to be a minimum, or at a minimum value, when it comes into the position pr.

80. *Several maxima possible.* These terms maxima and minima are not used to denote the greatest and least possible values which a function can assume, for the same function may admit of several maxima or several minima values, and some particular minimum may be greater than some particular maximum of the same function. Thus, in the annexed figure, PR and MN represent two maxima values of the variable ordinate; while pr, mn represent two minima values, and the minimum pr is greater than the maximum MN.



81. Besides the case represented by the figure in Art. 80, we may have the case represented by the annexed figure, where PR is a maximum ordinate, because it is greater than the ordinate immediately preceding or following; and pr is a minimum ordinate, because it is less than the preceding or following ordinate.



82. *How to discover maxima and minima values of a function.* If we examine the figure Art. 79, we shall see that when x is increasing and y is approaching a maximum value as PR, the tangent to the curve makes an acute angle with the axis of X; that is, while approaching P from the left, $\frac{dy}{dx}$ is positive. At P the tangent becomes parallel with the axis of X; that is, $\frac{dy}{dx}$ becomes zero. Immediately after passing P, the

tangent to the curve makes an obtuse angle with the axis of X; that is, $\frac{dy}{dx}$ is negative.

So, also, in approaching a minimum value as pr from the left, $\frac{dy}{dx}$ is negative; at the point p, $\frac{dy}{dx}$ is zero; and after passing p, $\frac{dy}{dx}$ is positive.

If we examine the figure in Art. 81, we shall see that while x increases and y is approaching a maximum value as PR, the tangent makes an acute angle with the axis of X; that is, $\frac{dy}{dx}$ is positive. At P the tangent in general becomes perpendicular to the axis of X; that is, $\frac{dy}{dx}$ becomes infinite; and after passing P, $\frac{dy}{dx}$ is negative.

Also, in approaching a minimum value as pr, $\frac{dy}{dx}$ is negative; at the point p, $\frac{dy}{dx}$ becomes infinite; and after passing p, $\frac{dy}{dx}$ is positive.

Hence we see that for increasing values of the variable, *when the function is a maximum the first differential coefficient changes its sign from + to −; and when the function is a minimum the first differential coefficient changes its sign from − to +.*

But a function which is continuous can change its sign only by becoming zero or infinity; hence, *for a maximum or minimum value of a function, the first differential coefficient must be either zero or infinity.*

If, therefore, we find the first differential coefficient of the function, and put it equal to zero and equal to infinity, the roots of these equations will give all the values of x which can possibly render the function a maximum or a minimum.

All the values of the variable thus found will not necessarily render the function a maximum or a minimum, because a variable quantity may pass through zero or infinity without changing its sign. Hence each value of the variable which

renders the first differential coefficient zero or infinity must be tested separately.

83. *Mode of testing the roots thus obtained.* Let a represent one of the roots of the equation $\frac{dy}{dx}=0$ or ∞. Substitute successively for x in the given function $a+h$ and $a-h$, where h is supposed to be a quantity indefinitely small. If both these results are less than that obtained by substituting a, the root will render the function a maximum; if both results are greater, the root will make the function a minimum.

If one result be greater and the other less than that obtained by substituting a, the root under consideration will give neither a maximum nor a minimum.

Or we may substitute $a+h$ and $a-h$ successively in the first differential coefficient. If the first result be + and the second −, the root will render the function a maximum; if the first be − and the second +, the root will render the function a minimum. If the two results have the same sign, this root will give neither a maximum nor a minimum.

The maximum or minimum value may be found by substituting the corresponding root in the given function.

Ex. 1. Find the value of x which will render u a maximum in the equation $u=10x-x^2$.

Differentiating, we obtain

$$\frac{du}{dx}=10-2x.$$

Putting this differential coefficient equal to zero, we have

$$10-2x=0,$$

whence $x=5$.

Let us now substitute $5-1$, 5, and $5+1$ successively for x in the given function.

Substituting 4 for x, we have $u_{,}=40-16=24$;

" 5 " " $u=50-25=25$;

" 6 " " $u'=60-36=24$.

Thus the function u appears to be a maximum when $x=5$.

This conclusion is not, however, a necessary inference from the above substitution, because the maximum value of the function might correspond to a value of x intermediate between 4 and 5 or 5 and 6. In order to remove any uncertainty of this kind, we may proceed as follows: For x substitute successively $5-h$ and $5+h$, and in each case we obtain $25-h^2$. Now, however small h may be, so long as it has a finite value, $25-h^2$ will be less than 25. Hence the value $x=5$ renders the function a maximum.

Or we may substitute $5-h$ and $5+h$ successively in the first differential coefficient. In the first case we obtain $+2h$, and in the second case $-2h$.

Ex. 2. Find the value of x which will render u a minimum in the equation $u=x^2-16x+70$. *Ans.* $x=8$.

Ex. 3. Find the value of x which will render u a minimum in the equation $u=4+(x-3)^2$. *Ans.* $x=3$.

84. *The first term of a series may be greater than the sum of all the others.* A more convenient method of determining the values of x which will render any proposed function a maximum or a minimum is founded upon the following principle:

If the variable x receive a small increment h, and a function of $x+h$ be expanded into a series according to the ascending powers of h, the quantity h may be assumed so small that any proposed term in the series shall be greater than the sum of all those that follow it.

Let $$u'=u+ph+qh^2+rh^3+ \text{ etc.}, \qquad (1)$$

where p, q, r, etc., are functions of x independent of h.

Equation (1) may be written

$$u'=u+h(p+qh+rh^2+ \text{ etc.});$$

or, putting $\mathrm{R}=q+rh+$ etc., we have

$$u'=u+h(p+\mathrm{R}h).$$

Now, if h be indefinitely diminished, $\mathrm{R}h$ will become less than p, which is independent of h; that is, p is greater than $\mathrm{R}h$, and consequently ph is greater than $\mathrm{R}h^2$, which is the sum of the succeeding terms of the series.

In the same manner it may be proved that h may be taken so small that qh^2 shall be greater than the sum of the succeeding terms. This conclusion does not apply to the failing cases of expansion by Taylor's theorem, some of the coefficients in those cases becoming infinite. See Art. 65.

85. *Sign of the second differential coefficient examined.* When the given function can be developed into a series by Taylor's theorem, the following method of finding maxima and minima will be most convenient:

Suppose we have $u=f(x)$,

and let the variable x be first increased by h and then diminished by h, and let

$$u'=f(x+h), \text{ and } u_{,}=f(x-h);$$

then, by Taylor's theorem, we shall have

$$u'-u=\frac{du}{dx}h+\frac{d^2u}{dx^2}\frac{h^2}{1.2}+\frac{d^3u}{dx^3}\frac{h^3}{1.2.3}+\text{ etc.}; \qquad (1)$$

$$u_{,}-u=-\frac{du}{dx}h+\frac{d^2u}{dx^2}\frac{h^2}{1.2}-\frac{d^3u}{dx^3}\frac{h^3}{1.2.3}+\text{ etc.} \qquad (2)$$

Now, if u be a maximum, it must be greater than either u' or $u_{,}$, however small h may be assumed, and we have seen, Art. 84, that h can be assumed so small that $\frac{du}{dx}h$ shall be greater than the sum of all the terms which follow it; hence, if $\frac{du}{dx}$ be either positive or negative, one of the quantities u', $u_{,}$ will be greater and the other less than u, which can not be the case if u be either a maximum or a minimum; and hence, when u is either a maximum or a minimum, we must have

$$\frac{du}{dx}=0.$$

If the first differential coefficient vanishes, we shall have

$$u'-u=\frac{d^2u}{dx^2}\frac{h^2}{1.2}+\frac{d^3u}{dx^3}\frac{h^3}{1.2.3}+\text{ etc.}; \qquad (3)$$

$$u_{,}-u=\frac{d^2u}{dx^2}\frac{h^2}{1.2}-\frac{d^3u}{dx^3}\frac{h^3}{1.2.3}+\text{ etc.}; \qquad (4)$$

and since h can be assumed of such a magnitude that $\frac{d^2u}{dx^2}\frac{h^2}{1.2}$ shall be greater than the sum of all the terms which follow it, and since h^2 is positive, the sign of the sum of each series will be the same as the sign of the second differential coefficient; that is, u will be a maximum if the value of x determined from the equation $\frac{du}{dx}=0$ renders the second differential coefficient negative, and a minimum if the second differential coefficient is positive.

If the second differential coefficient reduces to zero, the signs of equations (3) and (4) will again be opposite, and there can be neither a maximum nor a minimum unless the third differential coefficient reduces to zero, in which case the sign of the sum of each series will depend upon that of the fourth differential coefficient, and there will be a maximum when this is negative, and a minimum when it is positive, and so on.

If the first differential coefficient becomes infinite instead of zero, this is one of the cases in which Taylor's theorem fails (Art. 65), and the demonstration which is founded upon it fails also.

86. *General rule for maxima and minima.* Hence, to determine the value of x which will render a given function a maximum or a minimum, *find the first differential coefficient of the function, place it equal to* 0, *and find the roots of the equation thus formed. Substitute each of these roots in the second differential coefficient. Each one which gives a negative result will, when substituted in the function, make it a maximum; and each which gives a positive result will make it a minimum.*

If either root reduces the second differential coefficient to zero, substitute in the third, fourth, etc., until one is found which does not reduce to zero. If the differential coefficient which does not reduce to zero be of an odd order, this root will not render the function either a maximum or a mini-

num. But if it be of an even order, and negative, the root corresponds to a maximum; if positive, to a minimum.

Ex. 1. Find the values of x which will render u a maximum or a minimum in the equation

$$u = x^3 - 3x^2 - 24x + 85.$$

Differentiating, we obtain

$$\frac{du}{dx} = 3x^2 - 6x - 24.$$

Placing this equal to zero, we have

$$3x^2 - 6x - 24 = 0,$$

or

$$x^2 - 2x - 8 = 0,$$

the roots of which are $+4$ and -2.

The second differential coefficient is

$$\frac{d^2u}{dx^2} = 6x - 6.$$

Substituting 4 for x in the second differential coefficient, the result is $+18$, which, being positive, indicates a minimum; substituting -2 for x, the result is -18, which, being negative, indicates a maximum.

Hence the proposed function has a maximum value when $x = -2$, and a minimum value when $x = 4$.

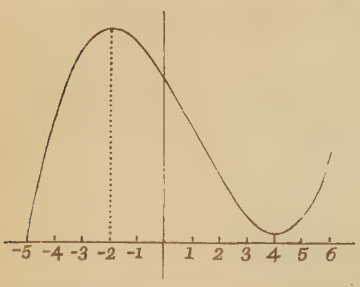
This result may be illustrated by assuming a series of values for x, and computing the corresponding values of u. Thus,

If $x = -4$, $u = 69$;
$x = -3$, $u = 103$;
$x = -2$, $u = 113$, maximum;
$x = -1$, $u = 105$;
$x = 0$, $u = 85$;
$x = +1$, $u = 59$;
$x = +2$, $u = 33$;
$x = +3$, $u = 13$;
$x = +4$, $u = 5$, minimum;
$x = +5$, $u = 15$;
$x = +6$, $u = 49$.

Thus it is seen that the value of the function increases while x increases from -4 to -2; it then decreases till $x = 4$;

and after that it increases again uninterruptedly, and will continue to do so, till x equals infinity. This peculiarity may be illustrated by a figure.

If we assume the different values of x to represent the abscissas of a curve, and erect ordinates equal to the corresponding values of u, the curve line which passes through the extremities of all the ordinates will be of the form represented in the annexed figure, where it is evident that the ordinates show a maximum corresponding to the abscissa -2, and a minimum corresponding to the abscissa 4.



It will be noticed that in constructing the preceding figure, and also in several similar cases, the unit adopted for the abscissas is not the same as the unit adopted for the ordinates.

Ex. 2. Find the values of x which will render u a maximum or a minimum in the equation

$$u=x^3-18x^2+96x-20.$$

Ans. $x=4$ renders the function a maximum, and $x=8$ renders it a minimum.

Ex. 3. Find the values of x which will render u a maximum or a minimum in the equation

$$u=x^3-18x^2+105x.$$

Ans. This function has a maximum value when $x=5$, and a minimum value when $x=7$.

Illustrate these results by a figure, as in the preceding example.

Ex. 4. Find the values of x which will render u a maximum or a minimum in the equation

$$u=x^4-16x^3+88x^2-192x+150.$$

The first differential coefficient gives us

$$x^3-12x^2+44x-48=0,$$

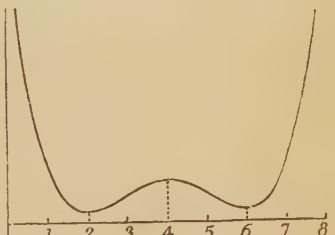
the roots of which equation are 2, 4, and 6. See Alg., Art. 464.

Substituting these values in the second differential coefficient, we find that the given function has a maximum value when $x=4$, and a minimum value when $x=2$ or 6.

If we assume a series of values for x, we shall obtain the corresponding values of u as follows:

If $x=1$, $u=31$;
$x=2$, $u=6$, minimum;
$x=3$, $u=15$;
$x=4$, $u=22$, maximum;
$x=5$, $u=15$;
$x=6$, $u=6$, minimum;
$x=7$, $u=31$;
$x=8$, $u=150$.

The curve representing these values has the form represented in the annexed figure, where two minima are seen corresponding to the abscissas 2 and 6, and a maximum corresponding to the abscissa 4.



Ex. 5. Find the values of x which will render u a maximum or a minimum in the equation

$$u=x^5-25x^4+\frac{700}{3}x^3-1000x^2+1920x-1100.$$

Ans. This function has two maximum values corresponding to $x=2$ and $x=6$, and two minimum values corresponding to $x=4$ and $x=8$.

87. *Abbreviations of the process of finding maxima and minima.* The process of finding maxima and minima may often be abridged by observing the following principles:

(1) *If the proposed function appears with a constant positive factor, such factor may be omitted* when seeking for those values of the variable which correspond to a maximum or a minimum.

Thus, let $$u = Ay,$$
where y is a function of x, and A is any positive constant. By differentiating we have

$$\frac{du}{dx} = \frac{A dy}{dx}; \quad \frac{d^2u}{dx^2} = \frac{A d^2y}{dx^2};$$

from which it appears that those values of x which make $\frac{dy}{dx} = 0$ will also make $\frac{du}{dx} = 0$, and conversely. Also, any of those values which make $\frac{d^2y}{dx^2}$ negative will also make $\frac{d^2u}{dx^2}$ negative; and any which make $\frac{d^2y}{dx^2}$ positive will make $\frac{d^2u}{dx^2}$ positive. Hence every value of x which will make y a maximum or a minimum will make Ay a maximum or a minimum.

(2) Whatever value of x renders y a maximum or a minimum must render y^n a maximum or a minimum, n being a positive integer.

Let $$u = y^n,$$
y being any function of x. Then

$$\frac{du}{dx} = \frac{ny^{n-1}dy}{dx};$$

$$\frac{d^2u}{dx^2} = ny^{n-1}\frac{d^2y}{dx^2} + n(n-1)y^{n-2}\frac{dy^2}{dx^2}.$$

Now every value of x which will make $\frac{dy}{dx} = 0$ will also make $\frac{du}{dx} = 0$. Also (since $\frac{dy^2}{dx^2} = 0$), if the same value makes ny^{n-1} *positive*, it will render the sign of $\frac{d^2u}{dx^2}$ the same as that of $\frac{d^2y}{dx^2}$; that is, if it makes y a maximum or a minimum, it will make u a maximum or a minimum. If it makes ny^{n-1} negative, it will give to $\frac{d^2u}{dx^2}$ a sign contrary to that of $\frac{d^2y}{dx^2}$; that is, if it makes y a maximum, it will make u a minimum, and conversely.

The converse of this proposition does not hold true, that all

the maxima and minima values of y^n belong also to y. For the equation

$$\frac{du}{dx}=ny^{n-1}\frac{dy}{dx}=0$$

may be satisfied by making either $y^{n-1}=0$, or $\frac{dy}{dx}=0$. Those values of x which satisfy the first but not the second of these equations may make y^n a maximum or a minimum, but will make y neither a maximum nor a minimum.

Hence, since a radical expression is involved to the corresponding power by omitting the radical sign, *in seeking for a maximum or a minimum of a function which is under a radical sign, the radical sign may be omitted.* Care must be taken, however, not to use any of those values which belong only to the power.

(3) In a similar manner it may be shown that in general any value of the variable which will render u a maximum or minimum will also render $\log. u$ and a^u a maximum or minimum; and also any value which will make u a maximum will make $\frac{1}{u}$, as also $-Au$, a minimum, and conversely.

Ex. 1. Find the values of x which will render u a maximum or a minimum in the equation

$$u=3x^3-x+c.$$

Ans. $x=\frac{1}{3}$ renders the function a minimum; and $x=-\frac{1}{3}$ renders it a maximum.

Ex. 2. Find the values of x which will render u a maximum or a minimum in the equation

$$u=x^3-9x^2+15x-3.$$

Ans. $x=1$ gives a maximum, and $x=5$ a minimum.

Ex. 3. Find the values of x which will render u a maximum or a minimum in the equation

$$u=\frac{x^2-7x+6}{x-10}.$$

Ans. $x=4$ gives a maximum, and $x=16$ a minimum.

Ex. 4. Find the values of x which will render u a maximum or a minimum in the equation

$$u=\frac{1-x+x^2}{1+x-x^2}.$$

Ans. $x=\frac{1}{2}$ gives a minimum.

Ex. 5. Find the values of x which will render u a maximum or a minimum in the equation

$$u=x^3-3x^2+6x+7.$$

Ans. This function has neither a maximum nor a minimum.

Ex. 6. Find the values of x which will render u a maximum or a minimum in the equation

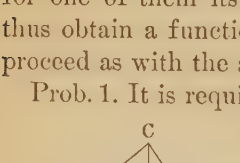
$$u=\sqrt[3]{x^3-3x^2}.$$

Ans. $x=0$ gives a maximum, and $x=2$ gives a minimum.

GEOMETRICAL PROBLEMS IN MAXIMA AND MINIMA.

88. *General directions.* In the solution of geometrical problems in maxima and minima, we must first obtain an algebraic expression for the quantity whose maximum or minimum state is required. This expression must be derived from the known properties of the figure in question, and will, of course, involve the variable or variables on which it depends. If the expression contains two dependent variables, we must find, from the known properties of the figure, or from the conditions of the problem, the relation between these variables, and substitute for one of them its value in terms of the other. We shall thus obtain a function of a single variable, and we may then proceed as with the algebraic problems of Art. 83, etc.

Prob. 1. It is required to find the maximum rectangle which can be inscribed in a given triangle.



Let b represent the base of the triangle ABC, h its altitude, and x the altitude of the inscribed rectangle. Then, by similar triangles, we have

$$CD : CG :: AB : EF,$$

or

$$h : h-x :: b : EF.$$

Hence $$EF=\frac{b}{h}(h-x).$$

The area of the rectangle is equal to EF × GD,

or $$\frac{b}{h}(hx-x^2),$$

which is to be a maximum.

But, since $\frac{b}{h}$ is a constant, the quantity $hx-x^2$ must also be a maximum (Art. 87).

Hence $$\frac{du}{dx}=h-2x=0,$$

or $$x=\frac{h}{2}.$$

Also, since $\frac{d^2u}{dx^2}=-2$, we know that the value $x=\frac{h}{2}$ gives a maximum value of the rectangle; that is, the altitude of the maximum rectangle is equal to half the altitude of the triangle.

Prob. 2. It is required to find the sides of the maximum rectangle which can be inscribed in a circle whose radius is r.

Ans. Each is equal to $r\sqrt{2}$.

Prob. 3. It is required to find the sides of the maximum rectangle which can be inscribed in a semicircle whose radius is r.

Ans. $r\sqrt{2}$, and $\frac{r}{\sqrt{2}}$.

Prob. 4. It is required to find the sides of the maximum rectangle which can be inscribed in an ellipse whose axes are $2a$ and $2b$.

The equation of an ellipse is

$$y=\frac{b}{a}\sqrt{a^2-x^2}.$$

The area of the rectangle

$$=2x\,.\,2y=\frac{4bx}{a}\sqrt{a^2-x^2}=\frac{4b}{a}\sqrt{a^2x^2-x^4}.$$

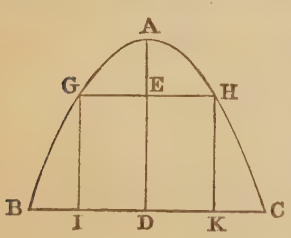
Hence $a^2x^2-x^4$ must be a maximum.

Ans. $a\sqrt{2}$, and $b\sqrt{2}$.

Prob. 5. It is required to find the sides of the maximum rectangle which can be inscribed in a semi-ellipse; 1st, when the bisecting line is the major axis; 2d, when it is the minor axis.

Ans. $a\sqrt{2}$, and $\frac{b}{\sqrt{2}}$; $\frac{a}{\sqrt{2}}$, and $b\sqrt{2}$.

Prob. 6. What is the altitude of the maximum rectangle which can be inscribed in a segment BAC of a parabola?



Put $AD=h$, and $AE=x$; then, by the equation of the parabola $y^2=4ax$, we have $GE^2=4ax$.

Hence

$$GE=2\sqrt{ax}, \text{ and } GH=4\sqrt{ax}.$$

Therefore, the area of GHKI is

$$4\sqrt{ax}(h-x),$$

which is a maximum.

or $x(h-x)^2$ is a maximum.

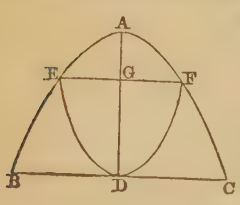
Hence $$x=\frac{h}{3}.$$

Or, the altitude of the maximum rectangle is two thirds of the axis of the parabola.

Prob. 7. Required the least square which can be inscribed in a given square.

Ans. Each angle of the required square is on the middle of a side of the given square.

Prob. 8. Within a given parabola BAC to inscribe the greatest parabola EDF, having its vertex D in the middle of the base BC.



Let $AD=h$, and $AG=x$; then

$$EG^2=4ax, \text{ and } EF=4\sqrt{ax}.$$

Therefore the area of the parabola

$$EDF=\frac{2}{3}EF\times GD,$$

$$=\frac{2}{3}\times 4\sqrt{ax}(h-x), \text{ a maximum};$$

or $x(h-x)^2$ is a maximum.

Hence $DG=\frac{2}{3}h$, *Ans.*

Prob. 9. What is the length of the axis of the maximum parabola which can be cut from a given right cone?

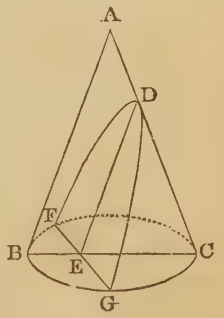
Put $BC=b$, $AB=h$, and $CE=x$; then $BE=b-x$.

Also (Geom., Bk. IV., Pr. 23),

$$FE=\sqrt{bx-x^2}, \text{ and } FG=2\sqrt{bx-x^2}.$$

By similar triangles we have

$$b:h::x:DE=\frac{hx}{b}.$$



Hence the area of the parabola is

$$\frac{2}{3}\cdot\frac{hx}{b}\cdot 2\sqrt{bx-x^2}, \text{ a maximum.}$$

Hence $x\sqrt{bx-x^2}$ is a maximum,

and $$x=\frac{3b}{4}.$$

Also, $$DE=\frac{hx}{b}=\frac{3h}{4};$$

that is, the axis of the maximum parabola is three fourths of the slant height of the cone.

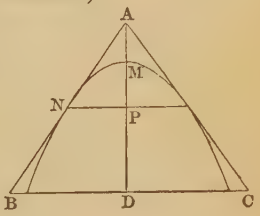
Prob. 10. It is required to circumscribe about a given parabola an isosceles triangle whose area shall be a minimum.

Put $x=MP=AM$; $y=NP$; $h=MD$;

then $y^2=4ax$, or $y=2\sqrt{ax}$.

Also $AP:NP::AD:BD$,

or $2x:y::x+h:BD=\frac{(x+h)2\sqrt{ax}}{2x}$.



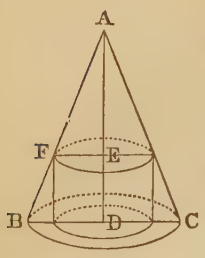
Area of

$$ABC=\frac{(x+h)^2\sqrt{ax}}{x}=\text{a minimum.}$$

Hence $$\frac{x^2+2xh+h^2}{\sqrt{x}}=x^{\frac{3}{2}}+2hx^{\frac{1}{2}}+h^2x^{-\frac{1}{2}}=\text{a minimum.}$$

Hence $$x=\frac{1}{3}h, \text{ and } AD=\frac{4h}{3};$$

that is, the altitude of the triangle is four thirds of the axis of the parabola.



Prob. 11. What is the altitude of a cylinder inscribed in a given right cone, when the volume of the cylinder is a maximum?

Let h represent the height of the cone, b the radius of its base, and x the altitude of the inscribed cylinder. Then, by similar triangles, we have AD : BD :: AE : EF,

or $$h : b :: h-x : \mathrm{EF} = \frac{b}{h}(h-x).$$

The area of the circle whose radius is EF is

$$\frac{\pi b^2}{h^2}(h-x)^2.$$

Multiplying this surface by DE, we obtain the volume of the cylinder

$$\frac{\pi b^2 x}{h^2}(h-x)^2, \text{ a maximum.}$$

Hence $x = \frac{h}{3}$; that is, the height of the greatest cylinder is one third of the altitude of the cone.

Prob. 12. It is required to determine the dimensions of a cylinder which shall have a given volume with the least possible surface.

Let c denote the volume of the cylinder, and x the radius of the base; the area of the base will be represented by πx^2.

Hence the height of the cylinder equals $\frac{c}{\pi x^2}$.

The convex surface of the cylinder is

$$\frac{c}{\pi x^2} \times 2\pi x = \frac{2c}{x}.$$

Hence the entire surface of the cylinder is

$$\frac{2c}{x} + 2\pi x^2 = \text{a minimum.}$$

Hence $$x = \left(\frac{c}{2\pi}\right)^{\frac{1}{3}},$$

and the height of the cylinder

$$= 2\left(\frac{c}{2\pi}\right)^{\frac{1}{3}};$$

that is, the height of the cylinder is equal to the diameter of the base.

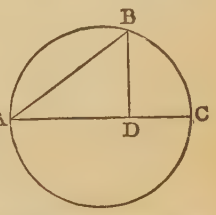
Prob. 13. It is required to determine the dimensions of a cylindrical vessel open at top which has the least surface with a given capacity.

Ans. The altitude of the cylinder is equal to the radius of the base.

Prob. 14. Required the altitude of the greatest cylinder which can be cut out of a sphere whose diameter is D.

Ans. $D\sqrt{\frac{1}{3}}$.

Prob. 15. Required the maximum cone which can be inscribed in a given sphere.



Let $AC=2r$, and $AD=x$.

Then $BD^2=AD\times DC=2rx-x^2$.

The solidity of the cone $=\frac{\pi x}{3}(2rx-x^2)$, a maximum.

Hence $x=\frac{4r}{3}$;

that is, the altitude of the cone whose volume is a maximum is four thirds of the radius of the sphere.

Prob. 16. Required the altitude of a cone inscribed in a given sphere which shall render the convex surface of the cone a maximum.

Ans. Four thirds of the radius of the sphere.

Prob. 17. It is required to determine the dimensions of a cone which shall have the greatest volume with a given amount of surface.

Let $x=$ the radius of the base, and $y=$ the slant height. The total surface of the cone $=\pi x^2+\pi xy=c$.

Hence $$y=\frac{c}{\pi x}-x.$$

Solidity of the cone $=\frac{1}{3}\pi x^2\sqrt{\frac{c^2}{\pi^2x^2}-\frac{2c}{\pi}}$, a maximum;

or $$\frac{cx^2}{\pi}-2x^4 \text{ is a maximum.}$$

Hence $$x=\frac{1}{2}\sqrt{\frac{c}{\pi}},$$

and $$y=\frac{3}{2}\sqrt{\frac{c}{\pi}};$$

that is, the slant height is three times the radius of the base.

Prob. 18. It is required to determine the dimensions of a cone whose convex surface is given ($=c$), and whose volume is a maximum.

Ans. The radius of the base $=\sqrt{\frac{c}{\pi\sqrt{3}}}$.

Prob. 19. It is required to determine the altitude of the least cone circumscribed about a given sphere.

Ans. Twice the diameter of the sphere.

Prob. 20. What is the altitude of the maximum cylinder which can be inscribed in a given paraboloid?

Ans. Half the axis of the paraboloid.

CHAPTER V.

TANGENTS AND NORMALS TO PLANE CURVES. ASYMPTOTES.

89. *Equation of a tangent line.* If x', y' represent the co-ordinates of a given point on a given curve, the equation of a straight line passing through this point will be (Anal. Geom., Art. 38)

$$y-y'=m(x-x'), \qquad (1)$$

where m represents the tangent of the angle which the line makes with the axis of abscissas. But we have seen, Art. 77, that the tangent of the angle which a tangent line at any point of a curve makes with the axis of abscissas is equal to the first differential coefficient of the ordinate of the curve; hence, in place of m in equation (1) we may substitute $\frac{dy'}{dx'}$, and we obtain

$$y-y'=\frac{dy'}{dx'}(x-x'), \qquad (2)$$

which is the equation of a tangent to a curve at a point whose co-ordinates are x', y'.

When it is required to find the equation of the tangent line to any curve, we must differentiate the equation of the curve, and find the value of $\frac{dy'}{dx'}$, which is to be substituted in equation (2).

Ex. 1. Let it be required to find the equation of the tangent line to a circle.

The equation of the circle is

$$y^2=r^2-x^2,$$

and by differentiation we find

$$\frac{dy'}{dx'}=-\frac{x'}{y'}.$$

Substituting this value in equation (2), we have for the equation of a tangent line

$$y-y'=-\frac{x'}{y'}(x-x'),$$

whence $$xx'+yy'=x'^2+y'^2=r^2,$$
which corresponds with Anal. Geom., Art. 73.

Ex. 2. Find the equation of the tangent to an ellipse.

The equation of an ellipse is
$$a^2y^2=a^2b^2-b^2x^2,$$
whence $$\frac{dy'}{dx'}=-\frac{b^2x'}{a^2y'}.$$
Hence the equation of the tangent line is
$$y-y'=-\frac{b^2x'}{a^2y'}(x-x'),$$
or $$a^2yy'+b^2xx'=a^2y'^2+b^2x'^2=a^2b^2,$$
which corresponds with Anal. Geom., Art. 134.

Ex. 3. Find the equation of a tangent line to a parabola.

Ex. 4. Find the equation of a tangent to the curve represented by $$3y^2+2x^2=6$$
at the point $x=1$, and construct the figure. Is there more than one tangent?

90. *Equation of a normal line.* Since the normal is perpendicular to the tangent, the equation of a normal at the point x', y' is
$$y-y'=-\frac{1}{m}(x-x'), \qquad (1)$$
or $$y-y'=-\frac{dx'}{dy'}(x-x'). \qquad (2)$$

Ex. 1. What is the equation of a normal to the curve represented by $$3x^2+4y^2=12$$
at the point $x=1$?

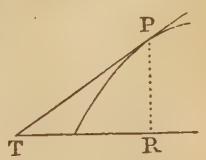
Ans. At $x=1$, $y=\pm\frac{3}{2}$. Hence there are two points indicated. The equation of the normal at the former is $y-2x+\frac{1}{2}=0$; and at the latter $y+2x-\frac{1}{2}=0$. Construct the figure and the two normals.

91. *The subtangent of any curve is the portion of the axis of abscissas intercepted between the tangent and the ordinate drawn from the point of tangency.*

If in equation (2), Art. 89, we make $y=0$, the corresponding value of $x'-x$, which is $y'\frac{dx'}{dy'}$,

will denote the length of the subtangent, where x', y' is the point of tangency.

The same result may be deduced from the annexed figure. Let PT be the tangent cutting the axis of abscissas in T. Then in the right-angled triangle PTR we have



$$\text{tang. T} : \text{PR} :: 1 : \text{TR};$$

that is,
$$\frac{dy}{dx} : y :: 1 : \text{TR} = \frac{ydx}{dy},$$

or, *the length of the subtangent to any point of a curve referred to rectangular axes is equal to the ordinate multiplied by the differential coefficient of the abscissa.*

92. *Length of the tangent.* In the right-angled triangle PTR we have
$$\text{PT}^2 = \text{PR}^2 + \text{RT}^2;$$

that is,
$$\text{PT}^2 = y^2 + y^2\frac{dx^2}{dy^2}.$$

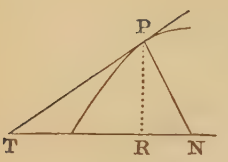
Hence the tangent $\text{PT} = \sqrt{y^2 + \frac{y^2dx^2}{dy^2}} = y\sqrt{1+\frac{dx^2}{dy^2}}$.

The same result may be expressed thus: *The length of the tangent to any point of a curve referred to rectangular axes is equal to the square root of the sum of the squares of the ordinate and subtangent.*

93. *The subnormal of any curve is the portion of the axis of abscissas intercepted between the normal and the ordinate drawn from the same point of the curve.*

If in equation (2), Art. 90, we make $y=0$, the corresponding value of $x-x'$, which is $y'\frac{dy'}{dx'}$,

will denote the length of the subnormal, where x', y' is the point of the curve from which the normal is drawn.



The same result may be deduced from the annexed figure. In the right-angled triangle PRN we have the proportion

$$1 : \text{PR} :: \text{tang. RPN} : \text{RN}.$$

But the angle RPN is equal to PTR; hence

$$1 : \text{PR} :: \text{tang. PTR} : \text{RN};$$

that is,

$$1 : y :: \frac{dy}{dx} : \text{RN} = y\frac{dy}{dx};$$

or, *the length of the subnormal to any point of a curve referred to rectangular axes is equal to the ordinate multiplied by the differential coefficient of the ordinate.*

94. *Length of the normal.* In the right-angled triangle PRN we have

$$\text{PN}^2 = \text{PR}^2 + \text{RN}^2;$$

that is,

$$\text{PN}^2 = y^2 + \frac{y^2dy^2}{dx^2};$$

Hence the normal $\text{PN} = \sqrt{y^2 + \frac{y^2dy^2}{dx^2}} = y\sqrt{1 + \frac{dy^2}{dx^2}}$.

The same result may be expressed thus: *The length of the normal to any point of a curve is equal to the square root of the sum of the squares of the ordinate and subnormal.*

95. *Tangent, etc., of a particular curve.* To apply these formulas to a particular curve, we must substitute in each of them the value of $\frac{dx}{dy}$ or $\frac{dy}{dx}$ obtained by differentiating the equation of the curve. The results thus obtained will be general for all points of the curve. If the values are required for a particular point of the curve, we must substitute in these results for x and y the co-ordinates of the given point.

Let it be required to apply these formulas to lines of the second order whose general equation is

$$y^2 = mx + nx^2.$$

Differentiating, we have

$$\frac{dy}{dx} = \frac{m + 2nx}{2y} = \frac{m + 2nx}{2\sqrt{mx + nx^2}}.$$

Substituting this value in the preceding formulas, we find

$$\text{the subtangent} = y\frac{dx}{dy} = \frac{2(mx+nx^2)}{m+2nx};$$

$$\text{the tangent} = \sqrt{y^2+\frac{y^2dx^2}{dy^2}} = \sqrt{mx+nx^2+4\left(\frac{mx+nx^2}{m+2nx}\right)^2};$$

$$\text{the subnormal} = y\frac{dy}{dx} = \frac{m+2nx}{2};$$

$$\text{the normal} = \sqrt{y^2+\frac{y^2dy^2}{dx^2}} = \sqrt{mx+nx^2+\tfrac{1}{4}(m+2nx)^2}.$$

96. *Tangent, etc., to a parabola.* By attributing proper values to m and n, the above formulas will be applicable to each of the conic sections. For the parabola $n=0$, and these expressions become,

the subtangent $= 2x$, which corresponds with Anal. Geom., Art. 93;

the tangent $= \sqrt{mx+4x^2}$;

the subnormal $= \frac{m}{2}$, which corresponds with Anal. Geom., Art. 98;

the normal $= \sqrt{mx+\frac{m^2}{4}}$.

97. *Subtangent, etc., of an ellipse.* In the case of the ellipse, these expressions assume a simpler form when the origin of coördinates is placed at the centre. The equation then becomes

$$a^2y^2+b^2x^2=a^2b^2; \qquad (1)$$

whence, by differentiating, we obtain

$$\frac{dy}{dx} = -\frac{b^2x}{a^2y}.$$

Hence, from Art. 91, we find

$$\text{the subtangent of the ellipse} = -\frac{a^2y^2}{b^2x}. \qquad (2)$$

But from equation (1) we have

$$\frac{a^2y^2}{b^2} = a^2-x^2.$$

Hence, from equation (2), we find

$$\text{the subtangent of the ellipse} = -\frac{a^2-x^2}{x},$$

which corresponds with Anal. Geom., Art. 135.

Also, from Art. 93 we find

$$\text{the subnormal of the ellipse} = \frac{ydy}{dx} = -\frac{b^2x}{a^2},$$

which corresponds with Anal. Geom., Art. 139.

98. *Subtangent, etc., for a circle.* In the case of the circle, a and b become equal, and we find

$$\text{the subtangent of the circle} = -\frac{y^2}{x};$$

$$\text{the tangent of the circle} = \sqrt{y^2+\frac{y^4}{x^2}} = \sqrt{\frac{r^2y^2}{x^2}} = \frac{ry}{x};$$

$$\text{the subnormal of the circle} = -x;$$

$$\text{the normal of the circle} = \sqrt{y^2+x^2} = r;$$

which results accord with well-known principles of Geometry.

Ex. 1. The latus rectum of a parabola is 4 inches, and an abscissa 9 inches; required the length of the corresponding ordinate and subtangent. *Ans.*

Ex. 2. The major axis of an ellipse is 30 inches, and the minor axis 16 inches; required the length of the subtangent corresponding to an abscissa of 10 inches, measured from the centre. *Ans.*

Ex. 3. The major axis of an ellipse is 6 inches, and the minor axis 4 inches; required the length of the subnormal corresponding to an abscissa of 2 inches, measured from the centre. *Ans.*

Ex. 4. The diameter of a circle is 10 inches; what is the length of the tangent and subtangent corresponding to an abscissa of 3 inches, measured from the centre?

Ans.

Ex. 5. What is the value of the subtangent and subnormal of the curve represented by $y^3 = ax^2$?

Ans. Subtangent $= 3x$; Subnormal $= \frac{a}{3y}$.

Ex. 6. Required the value of the subtangent of the logarithmic curve.

The equation of the logarithmic curve (Anal. Geom., Art. 245) is $x = \log. y.$

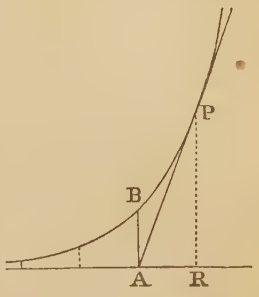
Differentiating, Art. 36, we obtain

$$dx = \frac{M dy}{y},$$

or

$$\frac{y dx}{dy} = M;$$

or, *the subtangent of the logarithmic curve is constant, and equal to the modulus of the system of logarithms employed.*



In the Naperian system M equals unity, and hence the subtangent AR is equal to unity, or AB. It is obvious that if the point of tangency be taken either to the right or left of P, as shown in the figure, the tangent line will not pass through the origin.

ASYMPTOTES TO CURVES.

99. *Branch of a curve.* Let x be the abscissa of any curve, and y the corresponding ordinate, and let

$$y = f(x)$$

be the equation of the curve. If we assign to x arbitrary values, and find the corresponding values of y, we shall determine different points of the curve, and by this means the curve can be constructed. If for each positive value of x the equation gives two values for y, this indicates that the curve has two branches, and if the values of y are all real when the assumed values of x increase indefinitely, the branches of the curve extend indefinitely; but if the values of y become imaginary, this indicates that the curve does not extend indefinitely on the positive side of the axis of ordinates. In the same manner, by assigning to x negative values, we may determine whether the curve has branches extending indefinitely on the negative side of the axis of ordinates. The parabola has two branches extending indefinitely on the positive side of

the axis of ordinates, but none on the negative side. The hyperbola has two branches extending indefinitely on either side of the axis of ordinates.

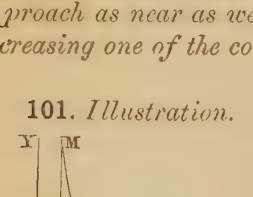
In Analytical Geometry, Part II., Section IX., are given the equations of numerous curves which have branches extending indefinitely in various directions.

Ex. 1. Determine whether the curve $y=x^3-x$ has an infinite branch, and construct the figure.

Ex. 2. Determine whether the curve $yx^3=81$ has an infinite branch, and construct the figure.

100. *Asymptotes.* Suppose a curve has one or more branches extending to an infinite distance from the origin, and that tangent lines are drawn at successive points of such a branch. Then, as we pass along the curve from point to point, we sometimes find that the *directions* of these tangents approach some definite limit, and also that the intercepts cut from the axes of co-ordinates tend to a finite limit. In such a case, there is a straight line toward which the successive tangents continually approach, and this straight line is called an *asymptote* to the curve. *An asymptote to a curve is the limiting position of the tangent when the point of contact moves to an infinite distance from the origin;* or we may define an asymptote to be *a line which will not meet the curve at any finite distance from the origin, but to which the curve may be made to approach as near as we please by continually increasing or decreasing one of the co-ordinates.*

101. *Illustration.*



Suppose a series of ordinates, BM, CN, DO, etc., to be drawn at equal distances from each other, and let each succeeding ordinate be half of the preceding. The curve which passes through the extremities of all these ordinates will continually approach the line AX, but will not meet it at any finite distance

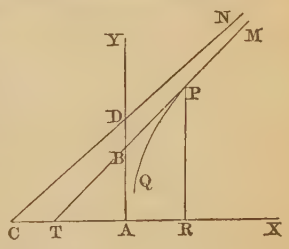
from A. The line AX is called an asymptote to the curve MNP, and is regarded as a tangent to the curve at an infinite distance from the origin.

Asymptotes may be either curvilinear or rectilinear. We shall consider only the latter.

102. *To determine whether a plane curve has rectilinear asymptotes.* First determine whether the curve has one or more branches extending to an infinite distance from the origin. If for all positive values of x from $+a$ to $+\infty$ one or more values of y are real, then for each set of real values of y the curve has an infinite branch extending to the right; if for all negative values of x from $-a$ to $-\infty$ one or more values of y are real, then for each set of real values of y the curve has an infinite branch extending to the left. If the curve has no infinite branch, it can not have an asymptote, since an asymptote is a tangent at an infinite distance from the origin.

If the curve has an infinite branch, we may proceed as follows to determine whether there is an asymptote to that branch:

Let MPQ be any plane curve, and let PT be a tangent to the curve at the point P whose co-ordinates are $AR=x'$ and $PR=y'$; and let PT meet the two axes in B and T. The equation of PT, Art. 89, is



$$y-y'=\frac{dy'}{dx'}(x-x').$$

If in this equation we make $y=0$, we obtain

$$x=AT=x'-\frac{y'dx'}{dy'}. \qquad (1)$$

If we make $x=0$, we obtain

$$y=AB=y'-\frac{x'dy'}{dx'}. \qquad (2)$$

Now, when the point of contact P recedes from the origin, the tangent PT approaches the asymptote NDC; and when P reaches an infinite distance, the tangent coincides with the asymptote, and AT becomes AC, and AB becomes AD.

If, when one or both of the co-ordinates x' and y' are indefinitely increased, the intercepts AD and AC are both finite, there will be an asymptote, whose position is determined by the two points D and C.

If one of these intercepts becomes infinite, and the other finite, there will be an asymptote parallel to that axis on which the intercept is infinite.

If both intercepts become infinite, there will be *no* asymptote.

If both intercepts become zero, the asymptote will pass through the origin of co-ordinates, and the tangent of the angle which it makes with the axis of X will be given by the expression $\frac{dy'}{dx'}$, when one or both of the co-ordinates become infinite.

Ex. 1. It is required to determine whether the hyperbola has asymptotes.

The equation of the hyperbola when the origin of co-ordinates is at the centre is (Anal. Geom., Art. 170)

$$y^2=\frac{b^2}{a^2}(x^2-a^2).$$

When x is greater than a, y has always two real values, however great x may be, and hence there are two infinite branches to the right.

Differentiating, we find

$$y\frac{dx}{dy}=\frac{a^2y^2}{b^2x}=\frac{x^2-a^2}{x}.$$

Therefore,
$$\mathrm{AT}=x'-\frac{y'dx'}{dy'}=\frac{a^2}{x'}.$$

Also,
$$\mathrm{AB}=y'-\frac{x'dy'}{dx'}=-\frac{b^2}{y'}.$$

When x is supposed to be infinite, the intercept $\mathrm{AT}=\frac{a^2}{x'}$ becomes zero. Also, when x is infinite, y must be infinite, and the intercept $\mathrm{AB}=-\frac{b^2}{y'}$ becomes zero.

Hence the hyperbola has two asymptotes, which pass through the origin, or the centre.

To determine the direction of the asymptotes, we have

$$\frac{dy}{dx}=\frac{b^2x}{a^2y}=\pm\frac{bx}{a\sqrt{x^2-a^2}}.$$

When $x=+\infty$, this expression $=\pm\frac{b}{a}$. Hence we see that the asymptotes are the produced diagonals of the rectangle described upon the axes.

In the direction of x negative there are also two infinite branches, which will be found to have the same lines through the origin as asymptotes. In fact, it will be found that every asymptote to an algebraic curve is an asymptote to two branches of the curve.

Ex. 2. It is required to determine whether the parabola has asymptotes.

The equation of the parabola is

$$y^2=4ax.$$

We see that there are two infinite branches to the right.

Differentiating, we find

$$y\frac{dx}{dy}=\frac{y^2}{2a}=2x.$$

Hence

$$\mathrm{AT}=x-\frac{ydx}{dy}=-x.$$

Also,

$$\mathrm{AB}=y-\frac{xdy}{dx}=\frac{y}{2}.$$

When x is supposed to be infinite, $\mathrm{AT}=-\infty$; and when y is infinite, $\mathrm{AB}=+\infty$. Hence the parabola has no asymptotes.

The ellipse and circle can have no asymptote, since neither has an infinite branch. Hence the hyperbola is the only line of the second order which has asymptotes.

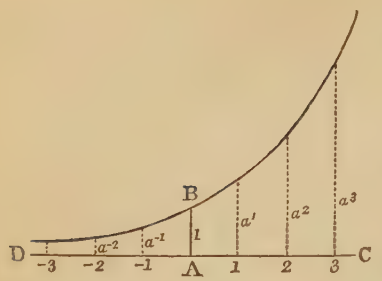
Ex. 3. It is required to determine whether the logarithmic curve has asymptotes.

The equation of the logarithmic curve is

$$x=\log. y,$$

or

$$y=a^x.$$



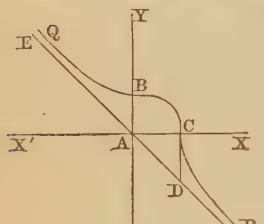
If x be taken infinite and negative, then

$$y=\frac{1}{a^{x}}=0;$$

that is, *the axis of abscissas is an asymptote to the curve.*

This curve has also an infinite branch extending to the right, to which there is no asymptote.

Ex. 4. It is required to determine whether the curve represented by $y^3=10-x^3$ has asymptotes.



This curve has an infinite branch extending to the right, and also one extending to the left.

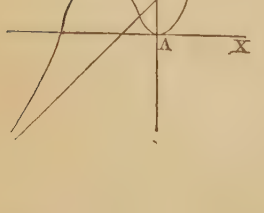
Differentiating, we find

$$\mathrm{AT}=\frac{10}{x^2},\ \text{and}\ \mathrm{AB}=\frac{10}{y^2},$$

each of which reduces to zero when x and y are infinite. Hence the curve has an asymptote passing through the origin.

The direction of the asymptote is given by the expression $\frac{dy}{dx}=-\frac{x^2}{y^2}=-\frac{(10-y^3)^{\frac{2}{3}}}{y^2}$, which equals -1 when y is infinite. Hence the asymptote makes an angle of 135° with the axis of X.

Ex. 5. It is required to determine whether the curve represented by $y^3=x^3+x^2$ has asymptotes.



Differentiating, we find

$$\mathrm{AT}=\frac{-x}{3x+2},$$

which $=-\frac{1}{3}$ when x is infinite.

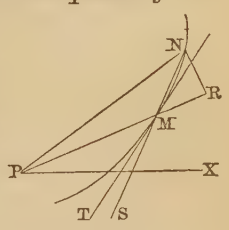
Also, $$\mathrm{AB}=\frac{x^2}{3(x^3+x^2)^{\frac{2}{3}}},$$

which $=\frac{1}{3}$ when x is infinite.

Hence there is an asymptote which cuts the axis of Y at $\frac{1}{3}$ above the origin, and the axis of X at $\frac{1}{3}$ on the left of the origin. The asymptote therefore makes an angle of 45° with the axis of X.

TANGENTS TO CURVES REFERRED TO POLAR CO-ORDINATES.

103. *To find the angle included between a tangent to a polar curve and the radius vector drawn to the point of contact.* Let P be the pole of the curve, and let the straight line MS cut the curve in two points, M and N. Represent the radius vector PM by r, PN by r', the angle XPM by θ, and the angle MPN by h. Also draw NR perpendicular to PM produced. Then we shall have



$$\text{tang. PMS} = \text{tang. NMR} = \frac{\text{NR}}{\text{MR}} = \frac{\text{NR}}{\text{PR} - \text{PM}} = \frac{\text{PN sin. } h}{\text{PN cos. } h - \text{PM}} =$$

$$\frac{r' \text{ sin. } h}{r' \text{ cos. } h - r}. \qquad (1)$$

But $\cos. h = 1 - 2(\sin. \frac{1}{2}h)^2$, and $\sin. h = 2 \sin. \frac{1}{2}h \cos. \frac{1}{2}h$

Hence

$$\text{tang. PMS} = \frac{r' \text{ sin. } h}{r' - r - 2r'(\sin. \frac{1}{2}h)^2} = \frac{r'}{\dfrac{r' - r}{\sin. h} - r' \text{ tang. } \frac{1}{2}h}. \qquad (2)$$

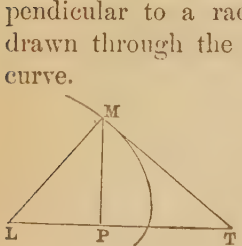
If now the point N be supposed to move toward M until it coincides with it, the secant line MS will come into the position of the tangent MT at M, and tang. PMS will become tang. PMT, which is therefore equal to the limiting value of the second member of equation (2), when h is indefinitely diminished. But the limit of $\frac{r' - r}{\sin. h}$ is $\frac{dr}{d\theta}$; and the limit of tang. $\frac{h}{2}$ is zero. Hence

$$\text{tang. PMT} = \frac{r}{\dfrac{dr}{d\theta}} = \frac{r d\theta}{dr};$$

or, *the tangent of the angle which a tangent line to the curve*

forms with the radius vector is equal to the radius vector multiplied by the differential coefficient of the variable angle.

104. A *polar subtangent* is a line drawn from the pole perpendicular to a radius vector, and limited by the tangent drawn through the point where the radius vector meets the curve.



Thus, if P be the pole of the curve, and MT be a tangent to the curve at the point M, then PT drawn perpendicular to PM, and meeting the tangent in T, is called the polar subtangent.

A *polar subnormal* is a line drawn from the pole perpendicular to a radius vector, and limited by the normal drawn through the point where the radius vector meets the curve.

Thus, if ML be drawn perpendicular to the tangent MT, and meet the line LPT drawn perpendicular to MP in L, then PL is called the polar subnormal.

105. *To find the value of the subtangent, etc., to a curve referred to polar co-ordinates.* The subtangent PT = PM tang. PMT $= r\frac{rd\theta}{dr} = \frac{r^2 d\theta}{dr}$; or, the length of the subtangent to a curve referred to polar co-ordinates is equal to *the square of the radius vector multiplied by the differential coefficient of the variable angle.*

Tangent MT $= \sqrt{PM^2 + PT^2} = \sqrt{r^2 + \frac{r^4 d\theta^2}{dr^2}} = r\sqrt{1 + \frac{r^2 d\theta^2}{dr^2}}$;

or, the length of the tangent in polar co-ordinates is *the square root of the sum of the squares of the subtangent and radius vector.*

The subnormal PL = PM tang. PML = PM cotang. PMT $= \frac{rdr}{rd\theta} = \frac{dr}{d\theta}$; or, the subnormal to a polar curve is equal to *the differential coefficient of the radius vector.*

Ex. 1. Find the value of the subtangent and subnormal to the spiral of Archimedes.

The equation of the spiral of Archimedes is $r=\frac{b\theta}{2\pi}$; whence

$$\frac{d\theta}{dr}=\frac{2\pi}{b}.$$

Hence the subtangent $=\frac{r^2d\theta}{dr}=\frac{2\pi r^2}{b}=\frac{2\pi}{b}\times\frac{b^2\theta^2}{4\pi^2}=\frac{b\theta^2}{2\pi}$.

If the moving point is supposed to start from the pole, then at the end of one revolution of the radius vector $\theta=2\pi$, and the subtangent $=2b\pi$, which equals *the circumference of the circle described with the radius vector of the point of contact.*

If the tangent be drawn at the extremity of the arc generated in m revolutions, we shall have $\theta=2m\pi$, and the subtangent $=\frac{b\theta^2}{2\pi}=\frac{4bm^2\pi^2}{2\pi}=2bm^2\pi=m.2bm\pi$; or, the subtangent after m revolutions is equal to *m times the circumference of the circle described with the radius vector of the point of contact.*

The subnormal $=\frac{dr}{d\theta}=\frac{b}{2\pi}$.

Ex. 2. Find the value of the subtangent of the hyperbolic spiral.

The equation of this spiral is $r=\frac{a}{\theta}$;

whence $$\frac{d\theta}{dr}=-\frac{\theta^2}{a}.$$

Hence the subtangent $=\frac{r^2d\theta}{dr}=-\frac{r^2\theta^2}{a}=-a$;

that is, *in the hyperbolic spiral the subtangent is constant.*

Ex. 3. Find the value of the subnormal of the logarithmic spiral.

The equation of this spiral is $\theta=\log. r$; whence

$$\frac{d\theta}{dr}=\frac{\mathrm{M}}{r}.$$

Hence the subnormal $=\frac{dr}{d\theta}=\frac{r}{\mathrm{M}}$, where M represents the modulus of the system of logarithms employed. In the Naperian

system of logarithms, M equals unity; hence *in the Naperian logarithmic spiral the subnormal is always equal to the radius vector.*

We also have tang. $PMT = \frac{rd\theta}{dr} = M$; that is, *the tangent of the angle which the tangent line makes with the radius vector is constant, and is equal to the modulus of the system of logarithms employed.* In the Naperian logarithmic spiral the tangent line is *inclined* 45° *to the radius vector.*

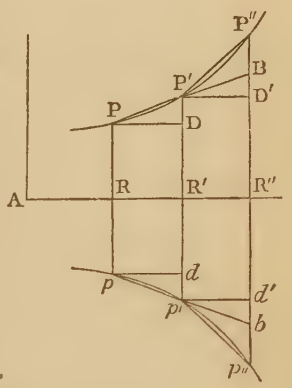
CHAPTER VI.

CONVEXITY AND CONCAVITY. SINGULAR POINTS OF CURVES. TRACING OF CURVES.

106. A curve is said to be *convex* at one of its points with respect to a given line, when in the immediate vicinity of that point the tangent lies between the curve and the given line. The curve is said to be *concave* when it lies between its tangent and the given line.

107. *To determine when a curve is convex toward the axis of abscissas.*

Let PP′P″ be a curve convex toward the axis of abscissas and above that axis, and let x and y be the coördinates of the point P. Let x be increased by any arbitrary increment RR′, which we will represent by h, and take R′R″ also equal to h. Draw the ordinates R′P′, R″P″; draw the secant line PP′, and produce it to B; join P′P″, and draw PD, P′D′ parallel to AR. We shall then have



$$RP=f(x)=y;$$

$$R'P'=f(x+h)=y+\frac{dy}{dx}h+\frac{d^2y}{dx^2}\frac{h^2}{2}+\text{ etc.} \qquad (1)$$

Also, $$R''P''=f(x+2h)=y+\frac{dy}{dx}2h+\frac{d^2y}{dx^2}\frac{4h^2}{2}+\text{ etc.} \qquad (2)$$

Hence $$DP'=R'P'-RP=\frac{dy}{dx}h+\frac{d^2y}{dx^2}\frac{h^2}{2}+\text{ etc.} \qquad (3)$$

Also, $$D'P''=R''P''-R'P'=\frac{dy}{dx}h+\frac{d^2y}{dx^2}\frac{3h^2}{2}+\text{ etc.} \qquad (4)$$

Subtracting equation (3) from equation (4), remembering that D′B is equal to DP′, we have

$$BP''=D'P''-DP'=\frac{d^2y}{dx^2}h^2+\text{ etc.} \qquad (5)$$

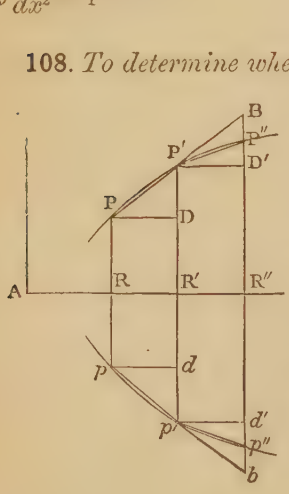
Now, Art. 84, the increment h can be taken so small that the sign of the second member of equation (5) will be the same as that of its first term; and since, by the definition of a curve convex toward the axis of x, the first member of the equation is positive, the second must also be positive; that is, the second differential coefficient is positive, and the ordinate, being situated above the axis of abscissas, is also positive.

If the curve is below the axis of abscissas, we shall have

$$bp''=d'p''-dp'=\frac{d^2y}{dx^2}h^2+\text{ etc.};$$

and since the first member of this equation is negative, the second will also be negative; that is, the second differential coefficient is negative. Hence we conclude that *if a curve is convex toward the axis of abscissas, the ordinate and second differential coefficient will have the same sign;* or, the product $y\frac{d^2y}{dx^2}$ is positive.

108. *To determine when a curve is concave toward the axis of abscissas.*



Let PP′P″ be a curve concave toward the axis of abscissas and above it, and let x and y be the co-ordinates of the point P. Let x be increased by any arbitrary increment RR′, which we will represent by h, and take R′R″ also equal to h. Draw the ordinates R′P′, R″P″; draw the secant line PP′, and produce it to B; join P′ and P″, and draw PD, P′D′ parallel to AR. We shall then have

$$RP=f(x)=y;$$

$$R'P' = f(x+h) = y + \frac{dy}{dx}h + \frac{d^2y}{dx^2}\frac{h^2}{2} + \text{ etc.} \qquad (1)$$

Also, $$R''P'' = f(x+2h) = y + \frac{dy}{dx}2h + \frac{d^2y}{dx^2}\frac{4h^2}{2} + \text{ etc.} \qquad (2)$$

Hence $$DP' = R'P' - RP = \frac{dy}{dx}h + \frac{d^2y}{dx^2}\frac{h^2}{2} + \text{ etc.} \qquad (3)$$

Also, $$D'P'' = R''P'' - R'P' = \frac{dy}{dx}h + \frac{d^2y}{dx^2}\frac{3h^2}{2} + \text{ etc.} \qquad (4)$$

Subtracting equation (3) from equation (4), we have

$$BP'' = D'P'' - DP' = \frac{d^2y}{dx^2}h^2 + \text{ etc.} \qquad (5)$$

Since the curve lies between the axis of x and the tangent, the first member of this equation is negative. Now let h be taken so small that the sign of the second member of equation (5) shall be the same as that of the first term; then $\frac{d^2y}{dx^2}$ must be negative; that is, the second differential coefficient will be negative, while the ordinate is positive.

If the curve is below the axis of abscissas, we shall have

$$bp'' = d'p'' - dp' = \frac{d^2y}{dx^2}h^2 + \text{ etc.},$$

where the second differential coefficient is positive, while the ordinate is negative. Hence we conclude that *if a curve is concave toward the axis of abscissas, the ordinate and second differential coefficient will have contrary signs;* or, the product $y\frac{d^2y}{dx^2}$ is negative.

A curve is evidently concave or convex toward the bottom of the page, according as $\frac{d^2y}{dx^2}$ is negative or positive. To determine the concavity or convexity toward any straight line whatever, change the co-ordinate axes, making that line the new axis of x, and find the sign of $y\frac{d^2y}{dx^2}$ in the new system.

Ex. 1. It is required to determine whether $x^2+y^2=r^2$ is convex or concave toward the axis of x.

Differentiating, we have

$$\frac{dy}{dx}=-\frac{x}{y}.$$

Also,
$$\frac{d^2y}{dx^2}=-\frac{x^2+y^2}{y^3}=-\frac{r^2}{y^3};$$

that is,
$$y\frac{d^2y}{dx^2}=-\frac{r^2}{y^2},$$

which is negative. Hence the curve is always concave toward the axis of x.

Ex. 2. It is required to determine whether $a^2y^2+b^2x^2=a^2b^2$ is convex or concave toward the axis of x.

Ex. 3. It is required to determine whether $y=5+3(x+2)^2$ is convex or concave toward the axis of x. *Ans.* Convex.

Ex. 4. It is required to determine whether $y=8x+6x^2-2x^3$ is convex or concave toward the axis of x.

Ans. From $x=-\infty$ to $x=-1$, convex;
" $x=-1$ to $x=0$, concave;
" $x=0$ to $x=1$, convex;
" $x=1$ to $x=4$, concave;
" $x=4$ to $x=\infty$, convex.

Ex. 5. It is required to determine whether $y=x^4-12x^3+48x^2-64x$ is convex or concave toward the axis of x.

Ans. From $x=-\infty$ to $x=0$, convex;
" $x=0$ to $x=2$, concave;
" $x=2$ to $x=\infty$, convex.

SINGULAR POINTS OF CURVES.

109. A singular point of a curve is one which is characterized by some peculiarity not belonging to other points of the curve. The most remarkable of these are, 1st, points of maxima and minima ordinates; 2d, points of inflection; 3d, multiple points; 4th, cusps; 5th, isolated or conjugate points; 6th, points of stopping.

Points of maxima and minima ordinates have already been

considered, Arts. 79 to 88. They differ from other singular points in that they depend upon the position of the axes, and change by transformation of co-ordinates.

110. *Points of inflection.* A point of inflection is a point at which, as x increases, a curve changes from convexity to concavity toward the bottom of the page, or conversely. Such a point is also characterized by the fact that the curve cuts its tangent at that point.

111. *To determine points of inflection.* When the curve is convex toward the bottom of the page, the second differential coefficient is positive; but when the curve is concave, it is negative. Hence at a point of inflection the second differential coefficient must change its sign. Therefore, between the positive and negative values there must be one value equal to zero, or infinity; and the roots of the equation

$$\frac{d^2y}{dx^2}=0, \text{ or } \frac{d^2y}{dx^2}=\infty,$$

will give the abscissas of the points of inflection.

But since a varying quantity does not necessarily change sign upon passing through zero or infinity, the points thus determined must be tested to ascertain whether the second differential coefficient does really change its sign. For this purpose, we may either construct the curve in the neighborhood of the point, or we may substitute in the second differential coefficient a value of x a little greater, and one a little less than the critical value, and observe whether the second differential coefficient has contrary signs for these new values of x.

Ex. 1. Determine whether the curve $y=a+(x-b)^3$ has a point of inflection.

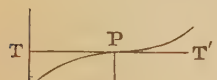
Differentiating, we find

$$\frac{dy}{dx}=3(x-b)^2,$$

and

$$\frac{d^2y}{dx^2}=6(x-b).$$

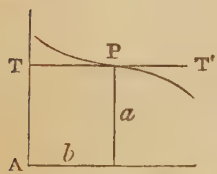
Placing this equal to zero, we have $x=b$, which in the equation of the curve gives $y=a$. To determine whether this point is a point of inflection, we find that, when $x<b$, the second differential coefficient is negative; but when $x>b$, the second differential co-efficient is positive. Therefore, there is an inflection of the curve at the point $y=a$, $x=b$.



On the left of P the curve falls below the tangent line TT', while on the right of P it runs above TT'.

Ex. 2. Determine whether the curve $y=a-(x-b)^3$ has a point of inflection.

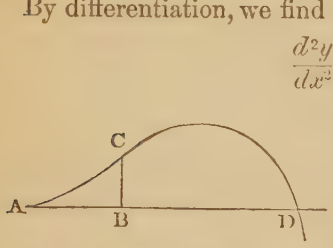
Ans. This curve is first convex, and then concave toward the axis of abscissas, and there is an inflection at the point $x=b$.



Ex. 3. Determine whether the curve $24y=3x+18x^2-2x^3$ has a point of inflection.

By differentiation, we find

$$\frac{d^2y}{dx^2}=\frac{3-x}{2}.$$



Putting this equal to zero, we obtain $x=3$, which gives a point of inflection. If x be <3, $3-x$ is positive; if x be >3, $3-x$ is negative.

Ex. 4. Determine whether the curve $y=ax+bx^2-cx^3$ has a point of inflection.

Ans. $x=\frac{b}{3c}$ gives an inflection.

Ex. 5. Determine whether the curve $y=x+36x^2-2x^3-x^4$ has a point of inflection.

Ans. There are inflections at $x=2$ and $x=-3$.

112. A *multiple* point is a point through which two or more branches of a curve pass. There are two species of multiple points: 1st, where two or more branches of a curve intersect;

2d, where two or more branches are tangent to each other. The latter are sometimes called *points of osculation.*

113. *To determine multiple points.* If the equation of a curve be of the nth degree, the equation may have n roots (Alg., Art. 436); that is, the ordinate y may have n different values. Since two or more branches of a curve pass through a multiple point, the different values of the ordinate y corresponding to the abscissa x for such a point become equal to each other, while for each value of x a little less or a little greater than the former the corresponding values of y are not equal. Having found a value of x for which two or more corresponding values of y are equal, while on each side of this point the corresponding values of y are not equal, determine the first differential coefficient $\frac{dy}{dx}$, and observe whether it admits of more than one value for this point. If it has two or more real values, the curve admits of more than one rectilinear tangent at this point; that is, two or more branches of the curve intersect, and the number of intersecting branches will be denoted by the number of real roots of $\frac{dy}{dx}$. Such a point is a multiple point of the first species.

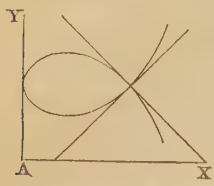
If the equation that gives the value of $\frac{dy}{dx}$ has two or more real and equal roots, different branches of the curve have a common tangent at this point, and this will be a multiple point of the second species.

Multiple points must be distinguished from points of a curve at which x is a maximum or minimum. Only one branch of a curve passes through a point of the latter description.

Ex. 1. Determine whether the curve $y=b\pm(x-a)\sqrt{x}$ has a multiple point.

We see that for every positive value of x there are generally two unequal values of y; but when $x=0$, the two values of y become $=b$; and when $x=a$, the two values of y become $=b$. On each side of the point $x=a$, $y=b$, y has two real values,

as we find by substituting for x, in the equation of the curve, $a+h$ and $a-h$ successively. This point is therefore a mul-



tiple point. The value of $\frac{dy}{dx}$ is

$$\pm\frac{3x-a}{2\sqrt{x}},$$

which for $x=a$ becomes $\frac{dy}{dx}=\pm\sqrt{a}$. Hence the tangents to the curve at this point make angles with the axis of x whose tangents are $+\sqrt{a}$ and $-\sqrt{a}$. This is therefore a multiple point of the first species.

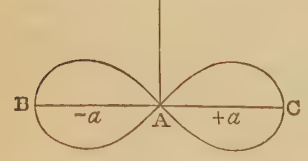
The point $x=0, y=b$ is *not* a multiple point, because when x is negative, y is imaginary, which shows that the curve does not extend to the left of the axis of y. There is therefore but one branch of the curve passing through this point, and this is simply a point where x is a minimum.

Ex. 2. Determine whether the curve $y^2=a^2x^2-x^4$ has a multiple point.

Extracting the root of each member, we have

$$y=\pm x(a^2-x^2)^{\frac{1}{2}},$$

from which we see that for each value of x between the limits



of $+a$ and $-a$ there are generally two values of y with contrary signs. Hence the curve has two branches which are symmetrical with respect to the axis of x. When $x=\pm a$, y has but one value, viz., 0; but these are not multiple points, because the curve does not extend beyond the limits $x=+a$ and $x=-a$. Also, when $x=0$, y has but one value, viz., 0, while on each side of this point y has two values. Here, then, is a multiple point. The value of $\frac{dy}{dx}$ is

$$\pm\frac{a^2-2x^2}{(a^2-x^2)^{\frac{1}{2}}},$$

which, when $x=0$, becomes $\frac{dy}{dx}=\pm a$. Hence one tangent line

makes an angle with the axis of abscissas whose tangent is $+a$; the other, an angle whose tangent is $-a$. This is therefore a multiple point of the first species.

Ex. 3. Determine whether the curve $ay^2=bx^2-x^3$ has a multiple point.

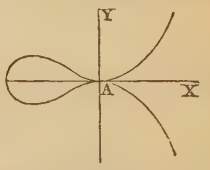
Ans. This curve has a multiple point of the first species at the origin; and the tangent of the angle which the tangent line makes with the axis of abscissas is $\pm\sqrt{\frac{b}{a}}$.

Ex. 4. Determine whether the curve $y^2=x^5+x^4$ has a multiple point.

The two critical points are $x=0$ and $x=-1$. The first differential coefficient is

$$\pm\frac{5x^2+4x}{2(x+1)^{\frac{1}{2}}},$$

which becomes, when $x=0$, $\frac{dy}{dx}=0$. The



two branches of the curve are tangents to each other at this point, and the origin is a multiple point of the second species.

114. A *cusp* is a point of a curve at which two branches have a common tangent, and stop at that point. If the two branches lie on *opposite* sides of the common tangent, the cusp is said to be of the *first species;* if both branches lie on the *same* side of the tangent, the cusp is said to be of the *second species.*

115. *To determine whether a curve has cusps.* Since a cusp is a variety of multiple points of the second species, the process is nearly the same as described in Art. 113; but since the branches stop at the cusp instead of running through it, the values of y on one side of this point are real, and on the other side imaginary. To distinguish a cusp from an ordinary multiple point, we must therefore compare the ordinate for that point with the ordinates of the curve on each side of that point.

If for any point y and $\frac{dy}{dx}$ have each but one value, and the values of y on one side of this point are imaginary, while $\frac{d^2y}{dx^2}$ has two values with different algebraic signs, this point is a cusp of the first species; if $\frac{d^2y}{dx^2}$ has two different values with the same algebraic sign, the point is a cusp of the second species.

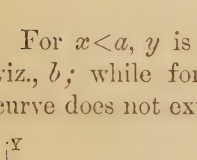
To ascertain the position of all the cusps that may belong to any curve whose equation is given, the process above described with respect to the axis of X should be repeated with respect to the axis of Y, x being regarded as a function of y, and $\frac{dx}{dy}$ being the first differential coefficient.

Ex. 1. Determine whether the curve $(y-b)^2=(x-a)^3$ has a cusp.

Extracting the square root, we have

$$y=b\pm(x-a)^{\frac{3}{2}}.$$

For $x<a$, y is imaginary; for $x=a$, y has but one value, viz., b; while for $x>a$, y has two real values. Hence the curve does not extend to the left of the point $x=a, y=b$; but on the right side of this point the curve extends indefinitely, and consists of two branches.



The value of $\frac{dy}{dx}$ is $\pm\frac{3}{2}(x-a)^{\frac{1}{2}}$, which becomes 0 when $x=a$. Hence the two branches at P have a common tangent, which is parallel to the axis of X, and there is a cusp. To determine the species of cusp, we find the value of the second differential coefficient for a value of x a little greater than a, as $a+h$.

The value of $\frac{d^2y}{dx^2}$ is $\pm\frac{3}{4}(x-a)^{-\frac{1}{2}}$, which becomes $\pm\frac{3}{4\sqrt{h}}$ when the abscissa is $a+h$. The cusp is therefore of the first species, as shown in the annexed diagram, where the upper branch, which is convex toward the axis of X, is pointed out by $+\frac{3}{4\sqrt{h}}$,

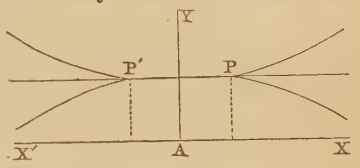
and the lower branch, which is concave, is pointed out by $-\frac{3}{4\sqrt{h}}$.

Ex. 2. Determine whether the curve $(y-b)^2=(x^2-a^2)^3$ has a cusp.

Extracting the square root, we have

$$y=b\pm(x^2-a^2)^{\frac{3}{2}}.$$

We find that y is imaginary for any value of x between $+a$ and $-a$; for $x=\pm a$, y has but one value; while for any point to the right of P or to the left of P′, y has two values. The curve has two cusps of the first species, one at P, the other at P′.



Ex. 3. Determine whether the curve $(y-x^2)^2=x^5$ has a cusp.

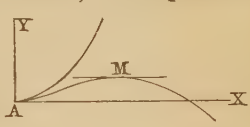
Extracting the square root, we have

$$y=x^2\pm x^{\frac{5}{2}}.$$

If x is negative, y is imaginary; for $x=0$, $y=0$; for every positive value of x, y has two real values, both of which are positive as long as $x^2>x^{\frac{5}{2}}$, or $x<1$; after which, one is positive and the other negative.

$$\frac{dy}{dx}=2x\pm\frac{5}{2}x^{\frac{3}{2}},$$

which $=0$ when $x=0$. It also equals 0 when $2x\pm\frac{5}{2}x^{\frac{3}{2}}=0$, or $x=\frac{16}{25}$. Hence the axis of X is tangent to both branches of the curve at the origin, and the tangent to the lower branch is parallel to the axis of X at the point M whose abscissa is $\frac{16}{25}$. For $x=0-h$ the values of y are impossible. The values of $\frac{d^2y}{dx^2}$ are $2\pm\frac{15}{4}x^{\frac{1}{2}}$, which become $2\pm\frac{15}{4}h^{\frac{1}{2}}$ when the abscissa is $0+h$, which are both positive when h is a very small magnitude. Hence at the origin there is a union of two branches of the curve, both convex toward the axis of X, forming a cusp of the second species.



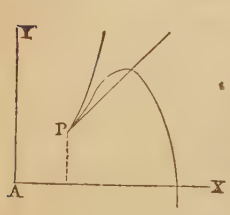
Ex. 4. Determine whether the curve $(y-x^2)^2=(x-a)^5$ has a cusp.

Extracting the square root, we have

$$y=x^2\pm(x-a)^{\frac{5}{2}}.$$

If $x<a$, y is imaginary; if $x=a$, $y=a^2$; if $x>a$, y has two real values, both of which are positive as long as $x^2>(x-a)^{\frac{5}{2}}$.

The values of $\frac{d^2y}{dx^2}$ are $2\pm\frac{15}{4}(x-a)^{\frac{1}{2}}$, which become $2\pm\frac{15}{4}h^{\frac{1}{2}}$ when the abscissa is $a+h$, which values are both positive when h is a very small magnitude. Hence at the point $x=a$ there is a cusp of the second species; and the tangent of the angle which a tangent line at this point makes with the axis of abscissas is $2a$.



116. *Definition.* A *conjugate* or *isolated* point is one whose co-ordinates satisfy the equation to the curve while the point itself is in general detached from every other in the curve.

For example, let $y^2=x^3-x^2$.

Here the values $x=0$, $y=0$ satisfy the equation to the curve; but no branch of the curve passes through the point thus determined, y being impossible for all negative values of x, and also for all positive values less than unity. Hence the origin of co-ordinates is a conjugate point in this curve.

117. *To determine whether a point is a conjugate point of a curve.* Let $y=f(x)$ be the equation of a curve which has a conjugate point; then, if this point is separated from every other in the curve, it follows that on each side of this point real values of one of the co-ordinates must give imaginary values for the other. Let the co-ordinates of the given point be a and b (where a and b are supposed to satisfy the equation of the curve), and let the abscissa a be increased by a small quantity, h; then the corresponding ordinate represented by $f(x+h)$ must be imaginary.

Now, by Taylor's theorem, Art. 64,

$$f(x+h)=y+\frac{dy}{dx}h+\frac{d^2y}{dx^2}\frac{h^2}{2}+\frac{d^3y}{dx^3}\frac{h^3}{2.3}+\text{ etc.}$$

Since y and h are both real, in order that $f(x+h)$ may be imaginary, it is necessary that one of the differential coefficients $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$, etc., should be imaginary. If, when $x=a$, either of these coefficients becomes imaginary, the corresponding point is a conjugate point. In order, therefore, to find the conjugate points of a curve, we examine the successive differential coefficients, to see if any one of them becomes imaginary for a particular value of x which satisfies the equation of the curve.

The labor of forming the higher orders of differential coefficients is often so great that, if the first differential coefficient does not become imaginary, it may be more convenient to substitute successively $a+h$ and $a-h$ for x in the equation of the curve. If both values of y thus found are imaginary, the point at which $x=a$ is a conjugate point.

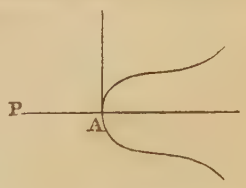
Ex. 1. Determine whether the curve $y^2=x(a+x)^2$ has a conjugate point.

The first differential coefficient $\frac{dy}{dx}=\pm\frac{a+3x}{2\sqrt{x}}$, which, when $x=-a$, becomes $\frac{\mp a}{\sqrt{-a}}$, an imaginary quantity, although $-a$ satisfies the equation of the curve. Hence the point P at which $x=-a, y=0$ is a conjugate point.

Extracting the square root of the given equation, we have

$$y=\pm(a+x)\sqrt{x},$$

from which we see that $y=0$ when $x=0$, and also when $x=-a$; but for any other value of x less than zero, y is imaginary. For every positive value of x, y has two real values, which shows that the curve has two branches extending indefinitely toward the right. The point P is therefore entirely isolated, and no



branch of the curve passes through it, although the co-ordinates of P satisfy the equation of the curve.

Ex. 2. Determine whether the curve $ay^2=x^2(x-b)$ has a conjugate point.

Ans. At the origin $\frac{dy}{dx}=\frac{\mp b}{\sqrt{-ab}}$, although the co-ordinates $x=0, y=0$ satisfy the equation of the curve. Hence the origin is a conjugate point.

TRACING OF CURVES.

118. To trace a curve is to discover from its equation its general form and leading peculiarities, and its position with reference to the assumed axes, so that we may draw a correct figure of the curve.

If it was possible to resolve an equation of any degree, we might follow the course of a curve represented by any algebraic equation by methods explained in Analytical Geometry. By ascribing to one of the variables different values, both positive and negative, we could determine at pleasure any number of points of the curve.

The Differential Calculus enables us to abridge this investigation, and may be employed even when the equation of the curve is of so high a degree that we are unable to obtain a general expression for one of the variables in terms of the other.

119. *To trace a plane curve from its equation referred to rectangular axes.* It is impossible to give specific directions for tracing the curve corresponding to every equation, for the methods which are most convenient in certain cases may not be the best for other cases.

1. The equation should, if possible, be solved with reference to one of the variables; that is, it should be put under the form $y=f(x)$.

2. Observe whether the curve is symmetrical with respect to either of the axes; that is, whether x or y, or both of them, enter into the equation only in even powers.

3. Observe whether the origin is a centre of the curve; that is, whether the equation remains unaltered when y is changed into $-y$, and x into $-x$. The centre of a curve is a point so situated that all chords of the curve drawn through that point are bisected in it.

4. Observe whether the curve passes through the origin; also, the shape of the curve near the origin.

5. Determine the points where the curve cuts the two axes, and find $\frac{dy}{dx}$ for those points.

6. Observe for what values of x, y is real, and for what values y is imaginary, and conversely. When there are several values of y, find for what values of x couples of them become imaginary. From the real values of y the number and position of the branches may be found. Remember that the degree of the equation of an algebraical curve determines the number of points in which a straight line may cut the curve.

7. Determine the limits of the curve in the direction of the axes, and ascertain if there are infinite branches by putting x or y equal to infinity.

8. Examine infinite branches for asymptotes, and determine them if they exist.

9. Observe whether for particular points of the curve the first differential coefficient of y reduces to $0, \infty, \frac{0}{0}$, an imaginary quantity, etc.

10. Determine the position of maxima and minima ordinates, and maxima and minima abscissas.

11. Find the second differential coefficient of y, and determine whether the curve is convex or concave, and find the points of inflection.

12. Having determined the number and position of the singular points, examine the course of the curve in their immediate vicinity.

Ex. 1. Trace the curve whose equation is $y = \frac{x}{1+x^2}$.

In order to illustrate the preceding directions, we will follow

them in succession as far as they are applicable to the present example.

(1) The equation is already reduced to the required form.

(2) The curve is not symmetrical with respect to either axis.

(3) The origin is a centre of the curve.

(4) and (5) The curve cuts the axis of x at the origin and at infinity, and the axis of y at the origin only. Near the origin (since x^2 is small), y is nearly equal to x, and numerically less than it. Therefore, near the origin, the curve lies between the axis of x and the straight line $y=x$.

(6) and (7) For each value of x there is one, and only one, value of y, and this has the same sign as x. There is therefore one branch extending from $x=-\infty$ to $x=+\infty$, and lying in the first and third quadrants.

Let AX and AY be the axes of x and y respectively, and A the origin of co-ordinates. Draw the line TAT′ whose equation is $y=x$. In the first quadrant the curve will lie below AT, and it will touch AT at the origin. In the third quadrant the curve will lie above AT′.

(8), (9), and (10) By differentiation, we find $\frac{dy}{dx}=\frac{1-x^2}{(1+x^2)^2}$.

Now $\frac{dy}{dx}=0$ when $1-x^2=0$, and also when $1+x^2=\infty$.

When $1-x^2=0$, or $x=\pm 1$, $y=\pm\frac{1}{2}$, and we have a maximum ordinate at C, and a minimum ordinate at c.

When $1+x^2=\infty$, $y=0$, and the infinite branch has the axis of x for its asymptote.

(11) and (12) We find, also, $\frac{d^2y}{dx^2}=-\frac{2x(3-x^2)}{(1+x^2)^3}$.

This expression changes sign when $2x(3-x^2)$ changes sign; that is, when $x=0$, and when $x=\pm\sqrt{3}$. Hence there are points of inflection at the origin, and at P and p. From P to p the curve is concave to the axis of x; elsewhere it is convex.

Ex. 2. Trace the curve whose equation is $a^3y=x(x-b)^3$.

(2) and (3) The curve is not symmetrical, nor is the origin the centre of the curve.

(4) Developing the equation, we have

$$a^3y=x^4-3bx^3+3b^2x^2-b^3x.$$

Since there is no constant term, the curve passes through the origin. When x is small, the equation becomes very nearly $a^3y=-b^3x$, and this straight line is a tangent to the curve at the origin.

When x is large, the lower powers of x are not important, and the equation becomes very nearly $a^3y=x^4$.

(5), (6), and (7) For each value of x there is one, and only one, value of y, and therefore there is a single branch from $x=-\infty$ to $x=+\infty$.

The value of y is plus when x is minus, and also when $x>b$. It is minus from $x=0$ to $x=b$.

(9) and (10) Differentiating, we have

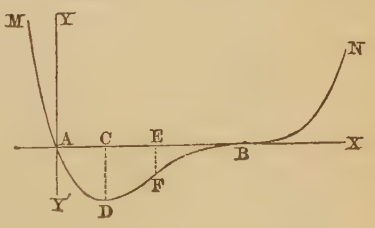
$$\frac{dy}{dx}=\frac{(x-b)^2(4x-b)}{a^3};$$

$$\frac{d^2y}{dx^2}=\frac{6(x-b)(2x-b)}{a^3};$$

$$\frac{d^3y}{dx^3}=\frac{24x-18b}{a^3}.$$

At the origin $\frac{dy}{dx}=\frac{-b^3}{a^3}$, and the line $a^3y=-b^3x$ is tangent to the curve, as shown above.

$\frac{dy}{dx}=0$ when $x=b$, and when $4x=b$. When $x=b$, $\frac{d^2y}{dx^2}=0$, and we must examine the value of $\frac{d^3y}{dx^3}$. This reduces to $\frac{6b}{a^3}$, and hence, Art. 86, $x=b$ does not give either maximum or minimum values of y. The second value, $x=\frac{b}{4}$, makes $\frac{d^2y}{dx^2}$ positive, and hence we have a minimum value of y at D.



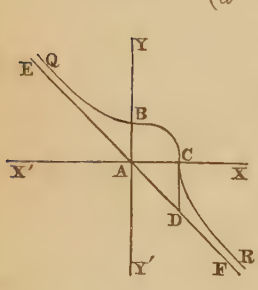
(11) The sign of $\frac{d^2y}{dx^2}$ changes from plus to minus at $x=\frac{b}{2}$, and from minus to plus at $x=b$. Hence, from F to B the curve is concave downward, and elsewhere it is convex; and F and B are points of inflection.

Ex. 3. Trace the curve $y^3=a^3-x^3$.

When $x=0$, $y=a$; take $AB=a$, and the curve passes through B. When $y=0$, $x=a$; take $AC=a$, and the curve passes through C. When x is infinite and positive, y is infinite and negative; therefore the arc CR is below AX, and extends indefinitely. When x is infinite and negative, y is infinite and positive; therefore the branch BQ extends indefinitely.

$\frac{dy}{dx}=-\frac{x^2}{(a^3-x^3)^{\frac{2}{3}}}$, which becomes infinite when $x=a$. Hence at C the curve cuts the axis of X at right angles.

$$\frac{d^2y}{dx^2}=-\frac{2a^3x}{(a^3-x^3)^{\frac{5}{3}}};\text{ and }\frac{d^3y}{dx^3}=-\frac{2a^3(4x^3+a^3)}{(a^3-x^3)^{\frac{8}{3}}}.$$



When $x=0$, $\frac{d^2y}{dx^2}=0$, and $\frac{d^3y}{dx^3}$ is not zero. Hence there is no maximum or minimum at B.

When x is $-$, $\frac{d^2y}{dx^2}$ is $+$;

when x is $+$ and $<a$, $\frac{d^2y}{dx^2}$ is $-$;

when x is $+$ and $>a$, $\frac{d^2y}{dx^2}$ is $+$.

Hence the arc QB is convex, BC concave, and CR convex to bottom of the page. When $x=0$ or a, $\frac{d^2y}{dx^2}$ changes sign, and therefore at B and C are points of inflection.

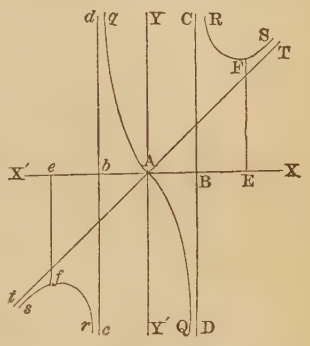
The equation to the asymptote EF is $y=-x$.

Ex. 4. Trace the curve $y=\frac{x(x^2+1)}{x^2-1}$.

For each value of x there is one, and only one, value of y. The curve is not symmetrical to either axis, but the origin is

the centre. Since $\frac{x^2+1}{x^2-1}$ is greater than 1 when x is greater than 1, y is in that case greater than x.

When $x=0$, $y=0$; therefore the curve passes through A. When $x<1$, y is negative, and the curve lies below AX. When $x=1$, y is infinite. Draw the ordinate BD, and produce it indefinitely; it will be an asymptote to the branch AQ.



When $x>1$, y is positive; when $x-1$ is very small, y is very great; and when $x=1$, y is infinite. Therefore BD produced is an asymptote, and the branch RFS is infinite.

When x is negative and less than 1, y is positive, and the curve lies above X′AX. When $x=-1$, y is infinite. Take $Ab=-1$, and the ordinate bd is an asymptote, the branch Aq being infinite. When $x-1$ is very small, y is very great; therefore bc produced is an asymptote to the branch rfs.

$\frac{dy}{dx}=\frac{x^2(x^2-4)-1}{(x^2-1)^2}$, which reduces to -1 when $x=0$; hence at the origin the curve cuts the axis of X at an angle of 135°.

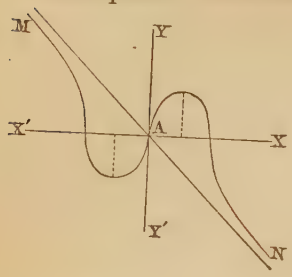
When $\frac{dy}{dx}=0$, $x=\pm\sqrt{2+\sqrt{5}}$; hence, if $AE=Ae=\sqrt{2+\sqrt{5}}$, the ordinates EF, ef will be minima.

$\frac{d^2y}{dx^2}=\frac{4x(x^2+3)}{(x^2-1)^3}$, which is $-$ when x is $+$ and <1, or $-$ and >1. It is also $+$ when x is $+$ and >1, or $-$ and <1; therefore every part of the curve is convex toward the axis of X.

$$\frac{d^3y}{dx^3}=-\frac{12(x^4+6x^2+1)}{(x^2-1)^4}.$$

When $x=0$, $\frac{d^2y}{dx^2}=0$, but $\frac{d^3y}{dx^3}=-12$; hence at A there is a point of inflexion.

The equation to the asymptote TAt is $y=x$.



Ex. 5. Trace the curve $y^3=a^2x-x^3$.

$$\frac{dy}{dx}=\frac{a^2-3x^2}{3(a^2x-x^3)^{\frac{2}{3}}};$$

$$\frac{d^2y}{dx^2}=-\frac{2a^2(3x^2+a^2)}{9(a^2x-x^3)^{\frac{5}{3}}}.$$

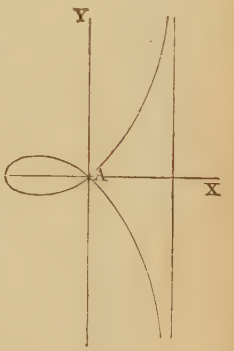
The equation to the asymptote is $y=-x$.

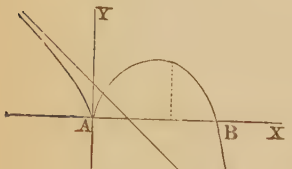
Ex. 6. Trace the curve $(a-x)y^2=(a+x)x^2$.

$$\frac{dy}{dx}=\pm\frac{a^2+ax-x^2}{a^2-x^2}\sqrt{\frac{a+x}{a-x}};$$

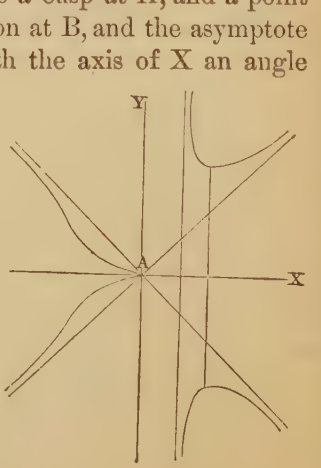
$$\frac{d^2y}{dx^2}=\pm\frac{a^2(2a+x)}{(a-x)(a^2-x^2)^{\frac{3}{2}}}.$$

The equation to the asymptote is $x=a$.



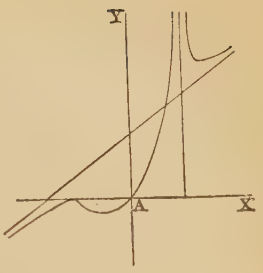


Ex. 7. Trace the curve $y^3=ax^2-x^3$. There is a cusp at A, and a point of inflection at B, and the asymptote makes with the axis of X an angle of 135°.



Ex. 8. Trace the curve $y^2=\frac{x^3}{x-a}$.

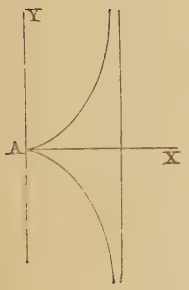
Ex. 9. Trace the curve $y=\frac{x(x+1)^2}{(x-1)^2}$.



$y=x+4$ is the equation to one asymptote.

$$\frac{dy}{dx}=\frac{x^3-3x^2-5x-1}{(x-1)^3};$$

$$\frac{d^2y}{dx^2}=\frac{8(2x+1)}{(x-1)^4}.$$



Ex. 10. Trace the curve $y^2(2a-x)=x^3$.

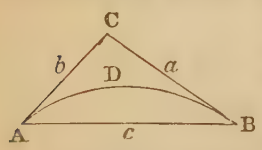
$$\frac{dy}{dx}=\frac{(3a-x)\sqrt{x}}{(2a-x)^{\frac{3}{2}}}.$$

$x=2a$ is the equation to the asymptote.

CHAPTER VII.

DIFFERENTIAL COEFFICIENT OF AN ARC, AREA, ETC.

120. *The limiting ratio of the length of an indefinitely small arc to its chord is one of equality.* Let ADB be an arc of any curve, and let it be taken so small that it is always concave toward its chord AB. At the extremities of this arc let the tangents AC, BC be drawn. The arc ADB is greater than the chord c, but less than the sum of the two tangents a and b (Chauvenet's Geom., Bk. V., Prop. 12).



By Trigonometry, Art. 49,

$$\frac{a}{c}=\frac{\sin.\,A}{\sin.\,C};\ \text{and}\ \frac{b}{c}=\frac{\sin.\,B}{\sin.\,C};$$

hence
$$\frac{a+b}{c}=\frac{\sin.\,A+\sin.\,B}{\sin.\,C}=\frac{\sin.\,A+\sin.\,B}{\sin.\,(A+B)}.$$

By Trigonometry, Art. 76,

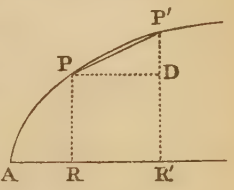
$$\frac{\sin.\,A+\sin.\,B}{\sin.\,(A+B)}=\frac{\cos.\tfrac{1}{2}(A-B)}{\cos.\tfrac{1}{2}(A+B)};$$

hence
$$\frac{a+b}{c}=\frac{\cos.\tfrac{1}{2}(A-B)}{\cos.\tfrac{1}{2}(A+B)}.$$

Let now the points A and B approach each other, and the arc ADB diminish continually. The angles A and B will decrease; and the arc may be taken so small that the angles A and B shall be less than any assigned quantity; therefore $A-B$ and $A+B$ both approach continually to zero; and $\cos.\tfrac{1}{2}(A-B)$ and $\cos.\tfrac{1}{2}(A+B)$ approach to unity, which is their common limit. Hence the limiting ratio of $a+b$ to c is a ratio of equality; and since the arc ADB can not be greater than $a+b$, nor less than c, much more is the limiting ratio of the arc to its chord a ratio of equality.

121. *The differential of an arc of any plane curve referred to rectangular co-ordinates is equal to the square root of the sum of the squares of the differentials of the co-ordinates.*

Let AR and RP be the co-ordinates x and y of any curve. If now we give to x any arbitrary increment h, and make $RR'=h$, the value of y will become equal to $P'R'$, which we will represent by y'. If we draw PD parallel to the axis AR', we shall have



$$\text{the chord } PP'=\sqrt{PD^2+P'D^2}=\sqrt{h^2+P'D^2}.$$

But $P'D=y'-y=\frac{dy}{dx}h+\frac{d^2y}{dx^2}\frac{h^2}{2}+$ other terms involving higher powers of h (Art. 64).

Substituting this value of P'D in the expression for the chord, we have

$$PP'=\sqrt{h^2+\frac{dy^2h^2}{dx^2}+\text{ etc.}};$$

$$=h\sqrt{1+\frac{dy^2}{dx^2}+\text{ etc.}}$$

Therefore $$\frac{PP'}{h}=\sqrt{1+\frac{dy^2}{dx^2}+\text{ etc.}},$$

which expresses the ratio of the chord PP' to the increment of the variable, and we must find the limit of this ratio by making the increment equal to zero, Art. 18.

But, by Art. 120, the limit of $\frac{\text{chord } PP'}{h}=$ the limit of $\frac{\text{arc } PP'}{h}$; and the terms omitted in the second member of the equation contain h, and therefore disappear. Hence, representing the length of the arc by L, we have

$$\frac{dL}{dx}=\sqrt{1+\frac{dy^2}{dx^2}};$$

or, multiplying by dx,

$$dL=\sqrt{dx^2+dy^2}.$$

Ex. 1. Find the differential of the arc of a circle.

From the equation $x^2+y^2=r^2$,

we find $xdx+ydy=0$; whence $dy=-\frac{xdx}{y}$.

Hence $$dL=\sqrt{dx^2+\frac{x^2dx^2}{y^2}}=\frac{dx}{y}\sqrt{x^2+y^2}.$$

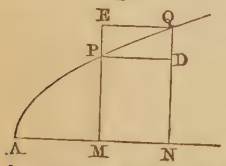
But $\sqrt{x^2+y^2}=r$, and $y=\sqrt{r^2-x^2}$.

Hence $$dL=\frac{rdx}{\sqrt{r^2-x^2}}.$$

Ex. 2. Find the differential of the arc of a parabola.

$$Ans.\ dL=\sqrt{\frac{a+x}{x}}.dx.$$

122. *The differential of any plane area included between a curve and the axis of x, and referred to rectangular co-ordinates, is equal to the ordinate into the differential of the abscissa.* Let APM be any plane area limited by a curve and the axis of x; and let x represent the abscissa AM, and y the corresponding ordinate PM. If we give to x an increment h, and make $MN=h$, the value of y will become QN, which we will represent by y'.



Through Q draw QE parallel to AN, meeting MP produced in E, and represent the area of APM by A. We shall then have incr. $x=h$, incr. $y=$DQ, and incr. A = area MPQN. Also,

$$\frac{\text{area MEQN}}{\text{area MPDN}}=\frac{EM.MN}{PM.MN}=\frac{EM}{PM}=\frac{y+\text{incr. }y}{y}=1+\frac{\text{incr. }y}{y}.$$

Now, as h approaches zero, incr. y also approaches zero; hence, if we take the limiting values of both sides of this equality, we shall have

$$\text{limit of }\frac{\text{area MEQN}}{\text{area MPDN}}=1.$$

But area MPQN is intermediate between the area MEQN and the area MPDN. Hence

$$\text{limit of }\frac{\text{area MPQN}}{\text{area MPDN}}=1.$$

Now area MPQN = incr. A, and area MPDN = $y.h$.

Hence $$\text{limit of } \frac{\text{incr. A}}{y.h} = 1;$$

or $$\frac{dA}{ydx} = 1.$$

Therefore $$\frac{dA}{dx} = y, \text{ or } dA = ydx.$$

In a similar manner, it may be proved that the differential of the area included between the curve and the axis of y is

$$dA = xdy.$$

Ex. 1. Find the differential of the area of a circular segment.

When the origin is at the centre, the equation is

$$y^2 = r^2 - x^2.$$

Hence $$dA = ydx = dx\sqrt{r^2 - x^2}.$$

When the origin is on the circumference, the equation is

$$y^2 = 2rx - x^2.$$

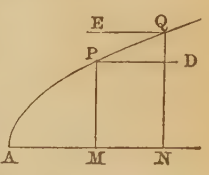
Hence $$dA = dx\sqrt{2rx - x^2}.$$

Ex. 2. Find the differential of the area of an ellipse.

Ans. $dA = dx\frac{b}{a}\sqrt{a^2 - x^2}$.

123. *The differential of the surface of a solid of revolution is equal to the circumference of a circle perpendicular to the axis multiplied by the differential of the arc of the generating curve.*

Let APQ be a curve which, by revolving about the axis of x, generates a solid of revolution. Let AM = x, and PM = y. If we give to x an increment h = MN, the value of y will become QN.



Let S = the surface generated by the revolution of the arc AP (= L) round the axis AN, and incr. S = the surface generated by the revolution of the arc PQ or incr. L, due to the increment MN given to x.

Draw PD and QE each equal to PQ, and each parallel to

AN. If PD revolve about AN, it will generate a cylinder whose surface is $2\pi PM.PD$. If QE revolve about AN, it will generate a cylinder whose surface is $2\pi QN.QE$. Hence

$$\frac{\text{surf. gen. by QE}}{\text{surf. gen. by PD}}=\frac{2\pi QN.QE}{2\pi PM.PD}=\frac{QN}{PM}=\frac{y+\text{incr. }y}{y}=1+\frac{\text{incr. }y}{y}.$$

Therefore limiting value of $\frac{\text{surf. gen. by QE}}{\text{surf. gen. by PD}}=1.$

But the surface generated by the arc PQ is intermediate between the surface generated by QE and the surface generated by PD; hence

$$\text{limiting value of } \frac{\text{surf. gen. by PQ}}{\text{surf. gen. by PD}}=1; \text{ that is,}$$

$$\text{limiting value of } \frac{\text{incr. S}}{2\pi y \text{ incr. L}}=1,$$

or

$$\text{limiting value of } \frac{\text{incr. S}}{\text{incr. L}}=2\pi y.$$

Hence

$$\frac{dS}{dL}=2\pi y; \text{ or } dS=2\pi y dL.$$

Substituting for dL its value, Art. 121, we have

$$dS=2\pi y\sqrt{dx^2+dy^2}.$$

If the curve revolve about the axis of y, we shall have

$$dS=2\pi x\sqrt{dx^2+dy^2}.$$

Ex. 1. Find the differential of the surface generated by the revolution of a parabola about its axis.

The equation of the parabola is $y^2=4ax$; whence

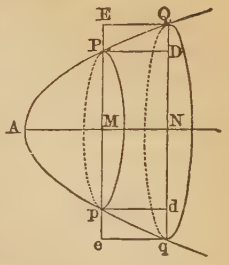
$$dx=\frac{ydy}{2a}.$$

Therefore $dS=2\pi y\sqrt{\frac{y^2dy^2}{4a^2}+dy^2}=\frac{\pi ydy}{a}\sqrt{y^2+4a^2}.$

Ex. 2. Find the differential of the surface of a sphere.

Ans. $dS=2\pi rdx.$

124. *The differential of the volume of a solid of revolution is equal to the area of a circle perpendicular to the axis multiplied by the differential of the abscissa of the generating*



curve. If the surface APM revolve around the axis of x, it will generate a solid of revolution. Put $AM=x$, and $PM=y$. If we give to x an increment $h=MN$, the value of y will become QN.

Let V = the volume of the solid generated by the revolution of APM around the axis AM. Then incr. $x=h$, incr. $y=DQ$, and incr. V = the solid generated by the revolution of MPQN. Now the rectangles PN and EN by their revolution will generate cylinders whose volumes may be found by Geom., Bk. X., Prop. 2. Hence

$$\frac{\text{vol. cyl. EQ}qe}{\text{vol. cyl. PD}dp}=\frac{\pi EM^2.MN}{\pi PM^2.MN}=\frac{EM^2}{PM^2}=\frac{(y+\text{incr.}y)^2}{y^2}=\left(1+\frac{\text{incr.}y}{y}\right)^2.$$

Now, as h approaches zero, incr. y also approaches zero; hence, if we take the limiting values of both sides of this equality, we shall have

$$\text{limit of}\ \frac{\text{vol. cyl. EQ}qe}{\text{vol. cyl. PD}dp}=1.$$

But the solid PQqp is always intermediate between the two cylinders; hence

$$\text{limit of}\ \frac{\text{vol. PQ}qp}{\text{vol. PD}dp}=1.$$

Now volume PQqp = increment V, and volume PD$dp=\pi y^2 h$. Therefore

$$\text{limit of}\ \frac{\text{incr. V}}{\pi y^2 h}=1,$$

or

$$\frac{dV}{\pi y^2 dx}=1;$$

that is,

$$\frac{dV}{dx}=\pi y^2, \text{ or } dV=\pi y^2 dx.$$

Ex. 1. Find the differential of the volume of a paraboloid of revolution.

The equation of the parabola is $y^2=4ax$; whence

$$dx=\frac{y\,dy}{2a}.$$

Therefore $$dV = \frac{\pi y^2 dy}{2a} = 4\pi a x dx.$$

Ex. 2. Find the differential of the volume of a prolate spheroid.

$$Ans.\ dV = \frac{\pi b^2}{a^2}(a^2 - x^2)dx.$$

POLAR CURVES.

125. *The differential of an arc of a curve referred to polar co-ordinates is equal to the square root of the sum of the squares of the differential of the radius vector, and of the product of the radius vector by the differential of the variable angle.*

Let P be the pole of the curve AMN; let the radius vector PM be represented by r, and the variable angle MPX by θ. Let the variable angle be increased by MPN, which we will represent by h; the value of r will become equal to PN, which we will represent by r'. Draw the chord MN. Then in the triangle PMN (Trig., Art. 78) we shall have

$$MN^2 = PM^2 + PN^2 - 2PM.PN \cos. MPN.$$

But $\cos. MPN = \cos. h = 1 - 2 \sin.^2 \frac{h}{2}$ (Trig., Art. 74).

Hence $$MN^2 = r^2 + r'^2 - 2rr'\left(1 - 2 \sin.^2 \frac{h}{2}\right);$$

$$= (r' - r)^2 + 4rr' \sin.^2 \frac{h}{2}.$$

Dividing by h^2, we have

$$\frac{MN^2}{h^2} = \frac{(r' - r)^2}{h^2} + rr'\left(\frac{\sin. \frac{h}{2}}{\frac{1}{2}h}\right)^2.$$

It is required now to find the limiting value of this ratio, as the point N is supposed to move toward M until it coincides with it.

The limit of $\frac{MN^2}{h^2}$ is $\frac{dL^2}{d\theta^2}$;

$$\text{the limit of } \frac{(r'-r)^2}{h^2} \text{ is } \frac{dr^2}{d\theta^2};$$

$$\text{and the limit of } rr'\left(\frac{\sin.\frac{h}{2}}{\frac{1}{2}h}\right)^2 \text{ is } r^2.$$

Therefore $\frac{dL^2}{d\theta^2}=\frac{dr^2}{d\theta^2}+r^2$; or $dL=\sqrt{dr^2+r^2d\theta^2}$.

Ex. 1. Find the differential of the arc of the logarithmic spiral.

The equation of the logarithmic spiral is $\theta=\log. r$.

Consequently $$d\theta=\frac{M dr}{r}.$$

Hence by substitution we find

$$dL=dr\sqrt{1+M^2}.$$

Ex. 2. Find the differential of the arc of the spiral of Archimedes.

Ans. $dL=\frac{dr}{a}\sqrt{a^2+r^2}$.

126. *The differential of the area included between the initial line of a polar curve, the arc of the curve, and a variable radius vector, is equal to the differential of the variable angle multiplied by half the square of the radius vector.*

Retaining the figure and notation of the last article, let the variable angle receive the increment MPN; then will the area APM receive the increment MPN. Now the triangle $PMN=\frac{1}{2}PM.PN \sin. MPN=\frac{1}{2}rr' \sin. h$. Hence

$$\frac{\text{triangle PMN}}{h}=\frac{1}{2}rr'\frac{\sin. h}{h}.$$

It is now required to find the limiting value of this ratio, as the point N moves toward M until it coincides with it. The

$$\text{limit of } \frac{\text{triangle PMN}}{h}=\text{the limit of } \frac{\text{curv. area PMN}}{h}=\text{limit}$$

of $\frac{1}{2}rr'\frac{\sin. h}{h}=$limit of $\frac{1}{2}rr'$ (Art. 43).

Hence, representing the curvilinear area by A, we have

$$\frac{dA}{d\theta}=\frac{r^2}{2};$$

or
$$dA=\frac{r^2d\theta}{2}.$$

Ex. 1. Find the differential of the area of the spiral of Archimedes.

Ans. $dA=\frac{r^2dr}{2a}$.

Ex. 2. Find the differential of the area of the logarithmic spiral.

Ans. $dA=\frac{Mrdr}{2}$.

CHAPTER VIII.

CONTACT. CURVATURE. EVOLUTES AND INVOLUTES. THE CYCLOID.

127. *Different orders of contact.* Two or more curves are said to *touch* each other when they have a common point, and a common tangent at that point. Thus the curves mn, pq, rs, having the common point P, and the common tangent AB, are said to touch each other at P. When

three curves touch each other at the same point P, the curve pq, which passes between the two others in the immediate neighborhood of P, is regarded as having a closer contact with the curve mn than the curve rs has. Geometers thus distinguish different orders of contact, and these orders of contact are determined by the differential coefficients of the equation of the curve.

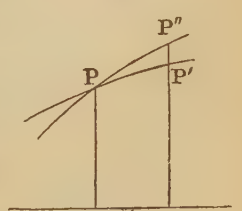
128. *How different orders of contact are distinguished.* Let there be two curves whose equations are $y=f(x)$ and $y'=\mathrm{F}(x')$; let them be referred to the same rectangular axes; and let their abscissas be increased by a common increment h. We shall then have (Art. 64)

$$f(x+h)=y+\frac{dy}{dx}h+\frac{d^2y}{dx^2}\frac{h^2}{2}+\frac{d^3y}{dx^3}\frac{h^3}{2.3}+\text{ etc.}, \qquad (1)$$

and

$$\mathrm{F}(x'+h)=y'+\frac{dy'}{dx}h+\frac{d^2y'}{dx'^2}\frac{h^2}{2}+\frac{d^3y'}{dx'^3}\frac{h^3}{2.3}+\text{ etc.} \qquad (2)$$

Then, if we suppose the curves to have the common point P, x will be equal to x' and y to y' for P. If the co-ordinates of this point be substituted in the preceding equations, $f(x+h)$ will represent the ordinate P′R′, and $\mathrm{F}(x'+h)$ will represent the ordinate P″R′. Also, $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$, etc.,



will represent the successive differential coefficients of the ordinate of the first curve, taken at the point P, and $\frac{dy'}{dx'}$, $\frac{d^2y'}{dx'^2}$, etc., the corresponding values for the second curve.

Since the first differential coefficient represents the tangent of the angle which a tangent line makes with the axis of abscissas, if we suppose not only that $x=x'$ and $y=y'$, but also that $\frac{dy}{dx}=\frac{dy'}{dx'}$, the two curves will have a common tangent at P. This is called a *contact of the first order*.

If we have at the same time $y=y'$, $\frac{dy}{dx}=\frac{dy'}{dx'}$, and $\frac{d^2y}{dx^2}=\frac{d^2y'}{dx'^2}$, the curves are said to have a *contact of the second order*.

If $y=y'$, $\frac{dy}{dx}=\frac{dy'}{dx'}$, $\frac{d^2y}{dx^2}=\frac{d^2y'}{dx'^2}$, and $\frac{d^3y}{dx^3}=\frac{d^3y'}{dx'^3}$, the curves are said to have a *contact of the third order*, and so on.

Now if all the terms of equation (1) are equal to the corresponding terms of equation (2), the two curves will coincide throughout, or be identical; and the greater the number of terms in the two developments which are equal, the closer will be the contact of the curves at P.

129. *Contact of the first order.* In order that a curve may have a contact of the nth order with a given curve, it appears from Art. 128 that $n+1$ equations must be satisfied. Hence, if the equation to a species of curves contains $n+1$ constants, we may, by assigning proper values to these constants, find the particular curve of the species that has a contact of the nth order with a given curve at a given point.

For example, the equation to a straight line is of the form $y=mx+c$; and since there are two constants m and c, we may, by properly determining them, find the straight line which has a contact of the first order with a given curve at a given point.

Let the equation of the proposed curve be $y^2=16x$, and let $y'=mx'+c$ be the equation of the straight line; then we have $\frac{dy'}{dx'}=m$, and $\frac{d^2y'}{dx'^2}=0$.

At a point of contact of the first order we must have

$$x=x',\ y=y',\text{ and }\frac{dy}{dx}=\frac{dy'}{dx'};$$

whence $$\frac{dy}{dx}=m,\text{ and }c=y-mx.$$

For the point at which $x=1$, $y=4$, $\frac{dy}{dx}=m=\frac{16}{2y}=2$, and $c=y-2x=2$.

Hence $y=2x+2$ is the equation of the straight line which touches the curve $y^2=16x$ at the point $x=1, y=4$, and this contact is of the first order. In general, *a straight line can have no higher order of contact than the first.*

130. *Contact of the second order.* The general equation to a circle contains *three* constants, and hence a circle may be found which has a contact of the second order with a given curve at a given point.

Let it be required to find the equation of the circle which has a contact of the second order with the curve $y^2=16x$ at the point $x=1, y=4$.

Differentiating the given equation twice, we find

$$\frac{dy}{dx}=\frac{8}{y},\text{ which at the given point becomes }=2;$$

$$\frac{d^2y}{dx^2}=-\frac{64}{y^3},\text{ which at the given point becomes }=-1.$$

The most general equation of the circle is

$$(x-a)^2+(y-b)^2=r^2.$$

Differentiating twice, we have

$$(x-a)dx+(y-b)dy=0,\text{ whence }\frac{dy}{dx}=-\frac{x-a}{y-b};$$

$$dx^2+dy^2+(y-b)d^2y=0,\text{ whence }\frac{d^2y}{dx^2}=-\frac{1+\frac{dy^2}{dx^2}}{y-b}.$$

Hence for the given point we have the three equations

$$(1-a)^2+(4-b)^2=r^2;$$

$$2=-\frac{1-a}{4-b};$$

$$1=\frac{1+\frac{4}{}}{4-b};$$

whence $a=11, \quad b=-1, \quad r=5\sqrt{5}.$

Hence the equation of the required circle is $(x-11)^2+(y+1)^2=125$. In general, *a circle can have no higher order of contact than the second.*

131. *Osculating circle determined.* We may determine in a similar manner the general equation of the circle which has a contact of the second order with any given curve at a given point. Let the equation of the circle be $(x-a)^2+(y-b)^2=r^2$, where a and b are the co-ordinates of the centre, and r the radius of the circle.

Differentiating the equation of the circle, we have

$$x-a=-\frac{dy}{dx}(y-b), \tag{1}$$

and

$$y-b=-\frac{dx^2+dy^2}{d^2y}; \tag{2}$$

whence $r^2=(y-b)^2+(y-b)^2\frac{dy^2}{dx^2}=(y-b)^2\left(\frac{dx^2+dy^2}{dx^2}\right);$

or $$r^2=\left(\frac{dx^2+dy^2}{d^2y}\right)^2\left(\frac{dx^2+dy^2}{dx^2}\right)=\frac{(dx^2+dy^2)^3}{(dxd^2y)^2};$$

whence $$r=\pm\frac{(dx^2+dy^2)^{\frac{3}{2}}}{dxd^2y}, \tag{3}$$

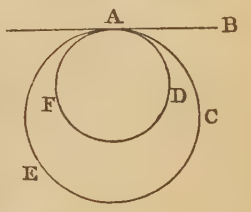
which is a general expression for the radius of the circle which has a contact of the second order with any given curve.

Curves which have a contact higher than the first order are said to *osculate;* and a circle which has a contact of the second order is called an *osculating circle.*

132. *Curvature of different curves compared.* The curvature of a plane curve is its deviation from a straight line; and of two curves, that which departs most rapidly from its tan-

gent is said to have the greatest curvature. Thus, of the two curves AC, AD, having the common tangent AB, the latter departs most rapidly from the tangent, and is said to have the greatest curvature.

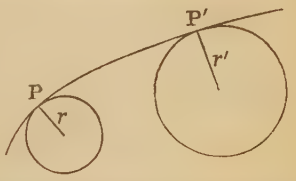
The curvature of the circumference of a circle is the same at all of its points, and also in all circumferences described with equal radii; but of two different circumferences, ACE, ADF, that one which has the least radius departs most rapidly from the tangent line. The curvature of a circle is *therefore usually measured by the reciprocal of its radius;* that is, in two different circles *the curvature varies inversely as their radii.*



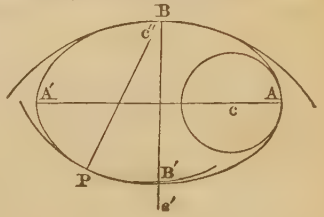
Since the osculating circle has a closer contact with a given curve than any other circle, its curvature is regarded as the same as that of the curve at the point of osculation. Hence, to compare the curvature at different points of a curve, we have only to compare the curvatures of the osculating circles drawn at those points. Thus, if r is the radius of the osculating circle at P, and r' that at P′, we shall have

$$\text{curv. at P} : \text{curv. at P}' :: \frac{1}{r} : \frac{1}{r'};$$

that is, *the curvature at different points of a curve varies inversely as the radius of the osculating circle.*



If an osculating circle be drawn to an ellipse at A, the vertex of its major axis, it will lie wholly within the ellipse; if an osculating circle be drawn at B, the extremity of the minor axis, its circumference will pass without the ellipse; at any other point, as P, it will cut the ellipse, as will be seen hereafter.



133. *Radius of curvature.* The radius of the osculating circle at a given point of a curve is called the radius of curvature at that point, and the centre of the osculating circle is the centre of curvature. A general expression for this radius is given in Art. 131. To find the radius of curvature for any particular curve, we must differentiate the equation of the curve twice, and substitute the values of dx, dy, and d^2y in the general expression for r. If the radius of curvature for a particular point of the curve is required, we must substitute for x and y the co-ordinates of the given point.

Since in determining the curvature of a curve, we require only the absolute length of the radius of curvature, we may employ either the + or − sign of the formula.

134. *To find the radius of curvature at any point of a conic section.* The general equation to a conic section (Anal. Geom., Art. 234) is $y^2 = mx + nx^2$;

whence $$dy = \frac{(m+2nx)dx}{2y},$$

and $$dx^2 + dy^2 = \frac{[4y^2 + (m+2nx)^2]dx^2}{4y^2}.$$

Also, $$d^2y = \frac{2ny dx^2 - (m+2nx)dxdy}{2y^2}$$

$$= \frac{[4ny^2 - (m+2nx)^2]dx^2}{4y^3} = -\frac{m^2dx^2}{4y^3}.$$

Substituting these values in the equation $r = \frac{(dx^2+dy^2)^{\frac{3}{2}}}{dxd^2y}$, we obtain

$$r = \frac{4}{m^2}\{mx + nx^2 + \tfrac{1}{4}(m+2nx)^2\}^{\frac{3}{2}}. \qquad (1)$$

In this expression, $\frac{m^2}{4}$ is the square of half the latus rectum, and the other factor is the cube of that part of the normal intercepted between the curve and the axis of x (Art. 95); that is, the radius of curvature at any point of a conic section is equal to *the cube of the normal, divided by the square of half*

the latus rectum; and the radii at different points of the same conic section are to each other as the cubes of the corresponding normals.

If in equation (1) we make $x=0$, we have $r=\frac{m}{2}$; that is, *the radius of curvature at the vertex of the major axis of any conic section is equal to half the latus rectum*, which for the ellipse and hyperbola is $\frac{b^2}{a}$.

If it is required to find the radius of curvature at the vertex of the minor axis of an ellipse, we make $m=\frac{2b^2}{a}$, $n=-\frac{b^2}{a^2}$, and $x=a$, which gives, after reduction,

$$r=\frac{a^2}{b};$$

that is, the radius of curvature at the vertex of the minor axis of an ellipse *is equal to one half the parameter of that axis.*

In the case of the parabola, $n=0$, and the general value of the radius of curvature becomes

$$r=\frac{(m^2+4mx)^{\frac{3}{2}}}{2m^2}.$$

Ex. 1. What is the radius of curvature for a point on a parabola whose abscissa is 9 and ordinate 6, the origin being at the vertex? What are the co-ordinates of the centre of curvature? Construct the parabola, with the osculating circle in position.

Ex. 2. Find the radius of curvature at the vertices of the axes of the ellipse whose axes are 10 and 6.

Ex. 3. Find the radius of curvature for a point of the same ellipse at which $x=2$. Find also the centre of curvature, and construct the osculating circle.

Ex. 4. Find the radius of curvature for the logarithmic curve $x=\log. y$.

$$Ans.\ r=\frac{(m^2+y^2)^{\frac{3}{2}}}{my}.$$

Ex. 5. Find the radius of curvature at any point in the cubical parabola $y^3=ax$.

$$Ans.\ r=\frac{(9y^4+a^2)^{\frac{3}{2}}}{6a^2y}.$$

Ex. 6. Find the radius of curvature in the equilateral hyperbola, referred to its asymptotes, $xy=m^2$.

$$Ans.\ r=\frac{(x^2+y^2)^{\frac{3}{2}}}{2m^2}.$$

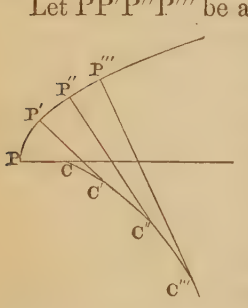
Ex. 7. Find the radius of curvature for the curve $y^4=a^3x$.

$$Ans.\ r=\frac{(16y^6+a^6)^{\frac{3}{2}}}{12a^6y^2}.$$

EVOLUTES AND INVOLUTES.

135. *Evolute and involute defined.* If osculating circles be drawn at *every* point of a curve, the curve line traced through their centres is called the *evolute* of that curve. Any curve, when considered with respect to its evolute, is called an *involute*, for reasons which will be hereafter explained.

Let PP′P″P‴ be a plane curve, and let the centre of curvature for the point P be C; then, as the point moves along the curve to P′, P″, etc., the centre of curvature will describe another curve, C, C′, C″, etc. The curve CC‴ thus described is called the evolute of the curve PP‴, and PP‴ is the involute of CC‴. Any number of points of the evolute may be determined by drawing normals at different points of the involute, and on each of these normals laying off the corresponding value of the radius of curvature as given in Art. 131.



136. *Given the equation of a plane curve, to find the equation of its evolute.* The equation of the evolute is the equation which expresses the relation between the centres of all the osculating circles of the involute.

The general equation of the circle is

$$(x-a)^2+(y-b)^2=r^2, \qquad (1)$$

where a and b denote the co-ordinates of the centre of the circle. To determine the equation of the evolute, we must find the relation of a to b in equation (1), regarding a and b as the co-ordinates of the centre of the circle of curvature, and consequently the co-ordinates of the evolute.

We have found (Art. 131) for the circle of curvature

$$x-a=-\frac{dy}{dx}(y-b); \qquad (2)$$

$$y-b=-\frac{dx^2+dy^2}{d^2y}; \qquad (3)$$

and combining these equations with that of the involute curve, we may obtain an equation from which x and y are eliminated.

We must therefore differentiate the equation of the involute twice; deduce the values of dy and d^2y, and substitute them in equations (2) and (3); two new equations will thus be obtained involving a, b, x, and y.

Combine these equations with the equation of the involute; and eliminate x and y; the resulting equation will contain only a and b and constants, and will be the equation of the evolute curve.

137. *To find the equation of the evolute of the common parabola.*

The equation of the involute is $y^2=mx$.

Whence $\frac{dy}{dx}=\frac{m}{2y}$; and $\frac{d^2y}{dx^2}=-\frac{m^2}{4y^3}$.

Substituting these values in equations (3) and (2), Art. 136, and reducing, we have

$$y-b=\frac{4y^3}{m^2}+y; \text{ whence } b^2=\frac{16y^6}{m^4},$$

and

$$x-a=-\frac{2y^2}{m}-\frac{m}{2}.$$

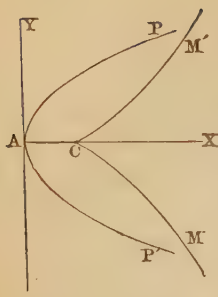
Substituting for y^2 its value mx, we have

$$b^2=\frac{16x^3}{m}; \text{ and } x-a=-2x-\frac{m}{2}; \text{ or } x=\frac{2a-m}{6}.$$

Substituting this value of x in the preceding equation, we have

$$b^2=\frac{2}{27m}(2a-m)^3,$$

which is the equation of the evolute, and shows it to be *the semicubical parabola* (Anal. Geom., Art. 237, Ex. 2).



If we make $b=0$, we have $a=\frac{m}{2}$, and hence the evolute meets the axis of abscissas at a distance AC from the origin equal to half the latus rectum. If we transfer the origin of co-ordinates from A to C, the above equation reduces to

$$b^2=\frac{16a^3}{27m}.$$

The evolute CM corresponds to the part AP of the involute, and CM′ to the part AP′.

138. *To find the equation of the evolute of the circle.*

The equation of the involute is $x^2+y^2=r^2$.

Whence $$\frac{dy}{dx}=-\frac{x}{y};\text{ and }\frac{d^2y}{dx^2}=-\frac{r^2}{y^3}.$$

Substituting these values in equations (3) and (2), Art. 136, and reducing, we find $a=0$, and $b=0$.

Hence the evolute is a point, viz., the centre of the circle. This is evidently correct, since all normals to a circumference meet at the centre of the circle.

139. *To find the equation of the evolute of the ellipse.*

The equation of the involute is $A^2y^2+B^2x^2=A^2B^2$.

Whence $$\frac{dy}{dx}=-\frac{B^2x}{A^2y};\text{ and }\frac{d^2y}{dx^2}=-\frac{B^4}{A^2y^3}.$$

Substituting these values in equations (3) and (2), Art. 136, and reducing, we find

$$y-b=\frac{y(A^4y^2+B^4x^2)}{A^2B^4},$$

and

$$x-a=\frac{x(A^4y^2+B^4x^2)}{A^4B^2}.$$

Now by the equation of the curve

$$A^2y^2=A^2B^2-B^2x^2,\text{ and }B^2x^2=A^2B^2-A^2y^2;$$

whence $A^4y^2+B^4x^2=B^2(A^4-C^2x^2)$, or $=A^2(B^4+C^2y^2)$, putting C^2 for A^2-B^2.

Hence, by substitution,

$$a=\frac{C^2x^3}{A^4}, \text{ and } b=-\frac{C^2y^3}{B^4};$$

whence $$x^2=\left(\frac{a^2A^8}{C^4}\right)^{\frac{1}{3}}, \text{ and } y^2=\left(\frac{b^2B^8}{C^4}\right)^{\frac{1}{3}}.$$

Substituting these values in the equation of the involute, we have

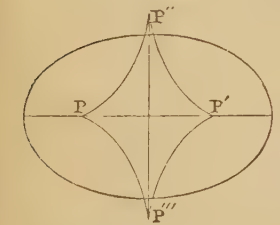
$$A^2\left(\frac{b^2B^8}{C^4}\right)^{\frac{1}{3}}+B^2\left(\frac{a^2A^8}{C^4}\right)^{\frac{1}{3}}=A^2B^2;$$

or, dividing each term by $\frac{A^2B^2}{C^{\frac{4}{3}}}$, we obtain

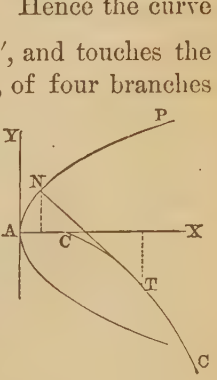
$$(bB)^{\frac{2}{3}}+(aA)^{\frac{2}{3}}=C^{\frac{4}{3}}=(A^2-B^2)^{\frac{2}{3}},$$

which is the equation of the evolute of an ellipse referred to its principal axes.

If $a=0$, then $b=\pm\frac{C^2}{B}$; so that the curve meets the axis of y in two points P″, P‴, equidistant from the origin. If $b=0$, then $a=\pm\frac{C^2}{A}$; so that the curve also meets the axis of x in two points, P, P′, equidistant from the centre.



If a is numerically greater than $\frac{C^2}{A}$, the ordinates become imaginary; and if b is numerically greater than $\frac{C^2}{B}$, the abscissas become imaginary. Hence the curve is limited by the four points P, P′, P″, P‴, and touches the axes at those points. It consists, therefore, of four branches symmetrically situated, as shown in the figure, with cusps at P, P′, P″, and P‴.



140. *A normal line to any plane curve is a tangent to its evolute.*

Let ANP be any curve, and CTC′ be its evolute. Let T be any point in the evolute whose co-ordinates are a and b; and from T draw a normal to the curve AP, meeting

the curve at N, whose co ordinates are x and y. The equation of this normal (Art. 90) becomes

$$y-b=-\frac{dx}{dy}(x-a);$$

or

$$x-a+\frac{dy}{dx}(y-b)=0. \qquad (1)$$

Now if the point N is moved along the curve, the corresponding point T will also be moved; that is, a and b, as well as y, are functions of x. In order to discover the change in each of these quantities caused by a change of x, we must differentiate equation (1), regarding all the quantities except dx as variables. We thus obtain

$$dx-da+\frac{d^2y}{dx}(y-b)+\frac{dy^2}{dx}-\frac{dbdy}{dx}=0;$$

or

$$dx^2+dy^2+yd^2y-bd^2y-dadx-dbdy=0. \qquad (2)$$

But since the point a, b, is on the evolute, we have (Art. 136)

$$y-b=-\frac{dx^2+dy^2}{d^2y};$$

whence

$$dx^2+dy^2+yd^2y-bd^2y=0.$$

Hence equation (2) becomes $-dadx-dbdy=0$; or

$$\frac{db}{da}=-\frac{dx}{dy}.$$

Hence equation (1), which is the equation of the normal to the involute at the point x, y, may be written

$$y-b=\frac{db}{da}(x-a),$$

which, by Art. 89, is the equation of a tangent to the evolute at the point a, b. Hence *the radius of curvature is normal to the involute and tangent to the evolute.*

141. *The difference between any two radii of curvature of a plane curve is equal to the arc of its evolute comprehended between them.*

Differentiating the equation

$$(x-a)^2+(y-b)^2=r^2,$$

regarding all the quantities as variable, we have

$$(x-a)dx+(y-b)dy-(x-a)da-(y-b)db=rdr.$$

But by Art. 140, $(x-a)dx+(y-b)dy=0$,

and $$y-b=(x-a)\frac{db}{da}.$$

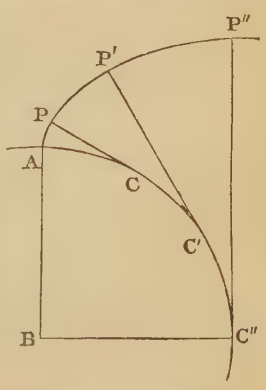
Hence $$(x-a)^2\left(\frac{da^2+db^2}{da^2}\right)=r^2, \qquad (1)$$

and $$-(x-a)\left(\frac{da^2+db^2}{da^2}\right)=\frac{rdr}{da}. \qquad (2)$$

Dividing equation (2) by the square root of equation (1), we have $$\sqrt{da^2+db^2}=dr.$$

But by Art. 121, $\sqrt{da^2+db^2}$ represents the differential of an arc of the evolute; hence *the differential of the radius of curvature of a curve is equal to the differential of the arc of its evolute;* or the radius of curvature and the length of the evolute vary by the same differences, so that the difference between any two radii is equal to the arc of the evolute comprehended between them.

142. *Involute described mechanically.* The preceding principles indicate how any involute may be described mechanically from its evolute. Let ACC′C″ be any curve, and let a thread be fastened to it at some point beyond C″. Let a pencil be fastened to the free end of the thread, and the thread drawn tight to the curve. Move the pencil along from A to P, P′, P″, etc., keeping the string tight as it is gradually unwound or *evolved* from the arc ACC′C″, and the pencil will describe a curve line APP′P″, which is the involute of ACC″; for at any point as P′ the portion of string unwound will coincide with the tangent at C′, and will therefore be the radius of curvature at the point P′



(Art. 140). From this mode of description the terms *evolute* and *involute* were derived.

look at.

THE CYCLOID.

143. A cycloid is the curve described by a point in the circumference of a circle, when the circle is rolled in the same plane along a given straight line.

The equation of the cycloid (Anal. Geom., Art. 243) is

$$x = \text{ver. sin.}^{-1} y - \sqrt{2ry - y^2}. \qquad (1)$$

144. *Differential equation of the cycloid.* The properties of the cycloid are most easily deduced from its differential equation, which is obtained by differentiating both members of equation (1).

By Arts. 45 and 53,

$$d.\,\text{ver. sin.}^{-1} y = \frac{rdy}{\sqrt{2ry - y^2}}.$$

By Art. 31, $$d\sqrt{2ry - y^2} = \frac{rdy - ydy}{\sqrt{2ry - y^2}}.$$

Hence $$dx = \frac{rdy}{\sqrt{2ry - y^2}} - \frac{rdy - ydy}{\sqrt{2ry - y^2}};$$

or, reducing, $$dx = \frac{ydy}{\sqrt{2ry - y^2}}, \qquad (2)$$

which is the differential equation of the cycloid.

145. *Subtangent, etc., of the cycloid.* If we substitute the preceding value of dx in the formulas of Arts. 91–94, we shall obtain the values of the tangent and subtangent, normal and subnormal of the cycloid. If $x = \text{AR}$, and $y = \text{PR}$, we shall have

$$\text{subtangent TR} = y\frac{dx}{dy} = \frac{y^2}{\sqrt{2ry - y^2}};$$

$$\text{tangent} \quad PT = y\sqrt{1+\frac{dx^2}{dy^2}} = \frac{y\sqrt{2ry}}{\sqrt{2ry-y^2}};$$

$$\text{subnormal} \quad RN = y\frac{dy}{dx} = \sqrt{2ry-y^2};$$

$$\text{normal} \quad PN = y\sqrt{1+\frac{dy^2}{dx^2}} = \sqrt{2ry}.$$

The subnormal RN is equal to PH of the generating circle, since each is equal to $\sqrt{2ry-y^2}$; hence the normal PN and the diameter EN intersect the base of the cycloid at the same point. Hence, if it is required to draw a normal to the cycloid at a given point, draw the generating circle passing through the given point, and from the point where this circle touches the base of the cycloid draw a straight line to the given point; it will be the normal required.

146. *Tangent parallel to base, etc.* From equation (2), Art. 144, we have

$$\frac{dy}{dx} = \frac{\sqrt{2ry-y^2}}{y} = \sqrt{\frac{2r}{y}-1}, \qquad (1)$$

which becomes 0 when $y=2r$, and ∞ when $y=0$; hence at the point B, the extremity of the greatest ordinate, the tangent is parallel to the base; and at the point A, where the curve meets the base, the tangent is perpendicular to the base.

147. *Curve concave toward the base.* Squaring both members of equation (1), Art. 146, we have

$$\frac{dy^2}{dx^2} = \frac{2r}{y}-1.$$

Differentiating, we have

$$\frac{2dyd^2y}{dx^2} = -\frac{2rdy}{y^2}, \text{ or } \frac{d^2y}{dx^2} = -\frac{r}{y^2}.$$

Since the second differential coefficient is always negative, and y always positive, the curve is always concave toward the axis of x (Art. 108).

148. *Radius of curvature.* Substituting the values of dy

and d^2y in the expression for the radius of curvature (Art. 131),

$$R=\frac{(dx^2+dy^2)^{\frac{3}{2}}}{dxd^2y},$$

we obtain

$$R=\frac{\left(\frac{2r}{y}\right)^{\frac{3}{2}}}{\frac{r}{y^2}}=2^{\frac{3}{2}}r^{\frac{1}{2}}y^{\frac{1}{2}}=2\sqrt{2ry}.$$

But we have found, Art. 145, the normal equal to $\sqrt{2ry}$; hence *the radius of curvature is equal to twice the normal at the point of osculation.*

If $y=0$, $R=0$; and if $y=2r$, $R=4r$. Hence the radius of curvature at A is equal to 0, and at B it is $4r$.

149. *Equation of the evolute.* To obtain the equation of the evolute, we must substitute the values of dy and d^2y already found for the cycloid in equations (3) and (2) of Art. 136. After reduction, we find

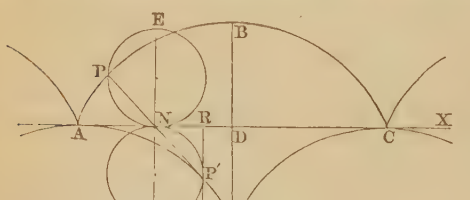
$$y-b=2y; \text{ whence } y=-b.$$

Also, $x-a=-2\sqrt{2ry-y^2}$; whence $x=a-2\sqrt{-2rb-b^2}$.

Substituting these values of x and y in the equation of the cycloid (Art. 143), we obtain

$$a=\text{ver. sin.}^{-1}(-b)+\sqrt{-2rb-b^2}, \qquad (1)$$

which is the transcendental equation of the evolute referred to the primitive origin and the primitive axes.



Produce BD to A′, making DA′ = BD, and transfer the origin to A′, the new axes being A′X′ and A′B, and the new coordinates a' and b'. Then for any point as P′ we shall have AR = a, RP′ = b, A′M = a', MP′ = b', a' being negative when measured to the left of A′.

Since $\quad AD=\pi r,\ a=\pi r+a';$

and since

$$RM=-2r,\ RP'=-2r+b',\ \text{or}\ b=-2r+b'.$$

Substituting these values in equation (1), we have

$$\pi r+a'=\text{ver. sin.}^{-1}(2r-b')+\sqrt{2rb'-b'^2}.$$

But $\quad \pi r=AD=\text{ver. sin.}^{-1}b'+\text{ver. sin.}^{-1}(2r-b').$

Hence $\quad -a'=\text{ver. sin}^{-1}b'-\sqrt{2rb'-b'^2},$

which is the equation of the evolute referred to the new axes. This equation is of the same form, and contains the same constants as the equation of the involute, except the sign of a'. But the branches A′A and A′C are symmetrical and equal, being derived from the equal cycloidal arcs AB and CB, and hence the sign of a' is indifferent; or *the evolute of a cycloid is an equal cycloid.*

The same conclusion might have been derived from the property proved in Art. 148, that the radius of curvature is double the normal. Since the centre of curvature for the point P is on the same line as the normal, and at a distance equal to twice PN, it follows that NP′=NP; and hence the arc NP of the circle whose radius is r is equal to the arc NP′ of a circle having the same radius, and touching AD in N. But the arc NP is equal to the line AN; therefore the arc LP′ is equal to A′L; or the curve A′P′A may be supposed to be generated by the point P′ as the circle LP′N rolls along LA′X′.

If we suppose the two circles NPE, NP′L to roll along the bases AX, A′X′ at the same rate, so as to keep their centres in the same vertical line, P′ will describe the evolute, while P describes the involute.

INTEGRAL CALCULUS.

CHAPTER I.

INTEGRATION OF MONOMIAL DIFFERENTIALS. OF CERTAIN BINOMIAL DIFFERENTIALS. DEFINITE INTEGRALS. TRIGONOMETRICAL FUNCTIONS.

150. *Integral Calculus defined.* In the preceding chapters we have shown how to differentiate a function of one or more variables. The integral calculus is the converse of the differential calculus, its object being to determine the expression or function from which a given differential has been derived. Thus we have found that the differential of x^n is $nx^{n-1}dx$. Therefore, if we have given $nx^{n-1}dx$, we know that it must have been derived from x^n, or x^n plus a constant term.

The primary object of the differential calculus has been defined (Art. 17) to be to determine the ratio between the rate of variation of the independent variable and that of the function into which it enters. The object of the integral calculus, therefore, is, *having given the ratio of the rate of variation of a function to that of the independent variable, to determine the function.*

151. *Integral defined.* The integral of a differential function is an expression which, being differentiated, produces the given differential.

Integration is the process of deducing the integral function from its differential.

152. *Symbol of integration.* The symbol by which the

operation of integration is represented is $\int$, a form derived from the old or long S.

Leibnitz regarded the differentials of functions as indefinitely small differences, and he regarded the function as being the *sum* of an infinite number of these infinitesimal portions; hence the integral of a differential expression was denoted by the letter S placed before the given expression. As it frequently became necessary to place the character S before very complicated expressions, the symbol was elongated into the form $\int$. On account of its convenience, this symbol has come into general use even by those who reject the philosophy of Leibnitz. When placed before a differential expression, it is to be regarded simply as denoting the expression from which the differential may be supposed to be derived. Thus

$$\int nx^{n-1}dx = x^n.$$

153. *Integration not always possible.* Whatever may be the form of a function, its differential may always be found by giving to the variable an arbitrary increment, finding the corresponding increment of the function, and determining the limiting value of this ratio. There is, however, no such general method of integration. If we are required to integrate $f(x)dx$, we can not be sure that the process is even possible; that is, we can not say that there is *any* function whose differential is $f(x)dx$, unless we find that this quantity may be obtained by differentiating some known function. The method which we are therefore compelled to pursue is to collect a large number of integrals by examining the results of the differentiation of simple functions, and then by various artifices to make the integration of other functions depend upon integrals already determined.

154. *Constant factor.* We have seen (Art. 19), that the differential of the product of a variable multiplied by a constant is equal to the constant multiplied by the differential of the variable. Hence we conclude that the integral of any dif

ferential multiplied by a constant quantity is equal to the constant multiplied by the integral of the differential.

Thus, since the differential of Ax is Adx, it follows that

$$\int Adx = Ax = A\int dx.$$

Hence, *if the expression to be integrated has a constant factor or divisor, this factor or divisor may be placed without the sign of integration* without affecting the value of the integral. Thus, $\int abx^3dx = ab\int x^3dx.$

155. *Sum of several differentials.* We have seen (Art. 22) that the differential of the algebraic sum of two or more functions is the algebraic sum of the differentials of those functions; that is,

$$d(ax^3 - 2a^2x^2 - x) = 3ax^2dx - 4a^2xdx - dx.$$

Hence $$\int(3ax^2dx - 4a^2xdx - dx) = ax^3 - 2a^2x^2 - x$$

$$= \int 3ax^2dx - \int 4a^2xdx - \int dx;$$

and generally,

$$\int(Adx + Bdx + \text{etc.}) = \int Adx + \int Bdx + \text{etc.};$$

that is, *the integral of the sum or difference of any number of differentials is equal to the sum or difference of their respective integrals.*

156. *Constant term added to the integral.* We have seen (Art. 20) that constant terms connected with any function of a variable disappear in differentiation; that is,

$$d(u + C) = du;$$

that is, an infinite number of expressions, differing from each other only in the value of the constant term or terms, when differentiated, will produce the same differential. Hence in integrating we represent any possible constant terms by an undetermined constant, which we usually represent by C. Thus

$$\int du = u + C;$$

that is, *a constant must always be added to the integral of a function.*

157. *Monomial differentials.* We have found (Art. 21) that the differential of ax^{m+1} is

$$a(m+1)x^m dx;$$

whence $$ax^m dx = \frac{ad(x^{m+1})}{m+1}.$$

Therefore $\frac{ax^{m+1}}{m+1}$ is the function whose differential is $ax^m dx$; that is,

$$\int ax^m dx = \frac{ax^{m+1}}{m+1} + C.$$

Hence, to obtain the integral of a monomial differential, *increase the primitive exponent of the variable by unity, multiply by the constant factor, if there is one, and divide the result by the new exponent.*

Ex. 1. What is the integral of $3ax^2dx$? *Ans.* $ax^3 + C$.

Ex. 2. The rate of variation of the independent variable x is to the rate of variation of a certain algebraic expression as 1 to $\frac{ax^2}{b}$; it is required to find that expression. *Ans.* $\frac{ax^3}{3b} + C$.

Ex. 3. What is the integral of $x^{\frac{1}{3}}dx$? *Ans.* $\frac{3}{4}x^{\frac{4}{3}} + C$.

Ex. 4. What is the integral of $\frac{dx}{\sqrt{x}}$, or $x^{-\frac{1}{2}}dx$? *Ans.* $2x^{\frac{1}{2}} + C$.

Ex. 5. What is the integral of $\frac{dx}{x^3}$, or $x^{-3}dx$? *Ans.* $-\frac{1}{2x^2} + C$.

Ex. 6. What is the integral of $ax^2dx + \frac{dx}{2\sqrt{x}}$? *Ans.* $\frac{ax^3}{3} + x^{\frac{1}{2}} + C$.

Ex. 7. What is the integral of $-\frac{3}{5}x^{-\frac{6}{5}}dx$? *Ans.* $3x^{-\frac{1}{5}} + C$.

Ex. 8. What is the integral of $ax^{\frac{2}{3}}dx$? *Ans.* $\frac{3ax^{\frac{5}{3}}}{5} + C$.

Ex. 9. What is the integral of $\frac{3x^{-\frac{2}{3}}dx}{a}$? *Ans.* $\frac{9x^{\frac{1}{3}}}{a} + C$.

Ex. 10. What is the integral of $\frac{x^{-\frac{m}{n}}dx}{a}$? *Ans.* $\frac{nx^{\frac{n-m}{n}}}{a(n-m)} + C$.

158. *Same method extended to polynomials.* We have seen (Art. 32) that any power of a polynomial may be differentiated by diminishing the exponent of the power by unity, and then multiplying by the primitive exponent, and by the differential of the polynomial. Thus the differential of $(ax+bx^2)^3$ is

$$3(ax+bx^2)^2(adx+2bxdx).$$

Hence, in order to integrate a compound expression consisting of any power of a polynomial multiplied by its differential, *increase the primitive exponent of the polynomial by unity, and divide the result by the new exponent.*

Ex. 1. What is the integral of $(a+3x^2)^2 6xdx$?

$$Ans. \frac{(a+3x^2)^3}{3}+C.$$

Ex. 2. What is the integral of $(2x^3-1)6x^2dx$?

$$Ans.$$

Ex. 3. What is the integral of $(6-x^3)^3x^2dx$?

$$Ans. -\frac{(6-x^3)^4}{12}+C.$$

Ex. 4. What is the integral of $(x+ax)^{\frac{1}{2}}(dx+adx)$?

$$Ans. \frac{2}{3}(x+ax)^{\frac{3}{2}}+C.$$

Ex. 5. What is the integral of $\frac{dx}{(1+x)^{\frac{1}{3}}}$? *Ans.*

Ex. 6. What is the integral of $(ax^2+bx^3)^2(2ax+3bx^2)dx$?

Ex. 7. What is the integral of $(ax+bx^2)^{\frac{1}{3}}(a+2bx)dx$?

Ex. 8. What is the integral of $\frac{dx}{(a-x)^5}$?

Ex. 9. What is the integral of $\frac{4xdx}{(1-x^2)^2}$?

Ex. 10. What is the integral of $\frac{x^8dx}{\sqrt{a^9+bx^9}}$?

159. *When the exponent is* -1. The rule given in Art. 157 fails when $m=-1$, for in this case we should have

$$\int x^{-1}dx=\frac{x^{-1+1}}{-1+1}=\frac{x^0}{0}=\frac{1}{0}=\infty,$$

which shows that the rule is inapplicable.

But $x^{-1}dx$ is the same as $\frac{dx}{x}$, and we know (Art. 37) that this expression was obtained by differentiating the logarithm of the denominator. Therefore

$$\int x^{-1}dx = \int \frac{dx}{x} = \log. x + C.$$

Also,

$$\int \frac{adx}{x} = a \log. x + C;$$

that is, if the numerator of a fraction is the product of the differential of its denominator by a constant quantity, *its integral is the product of the constant by the Naperian logarithm of the denominator.*

Ex. 1. What is the integral of $\frac{dx}{a+x}$? *Ans.* $\log.(a+x)+C.$

Ex. 2. What is the integral of $\frac{2bxdx}{a+bx^2}$?

Ans. $\log.(a+bx^2)+C.$

Ex. 3. What is the integral of $\frac{2bxdx}{a-bx^2}$?

Ans. $-\log.(a-bx^2)+C.$

Ex. 4. What is the integral of $\frac{adx}{bx}$? *Ans.*

Ex. 5. What is the integral of $\frac{ax^2dx}{x^3}$?

Ex. 6. What is the integral of $\frac{(2+2x)dx}{2x+x^2}$?

Ex. 7. What is the integral of $\frac{5x^3dx}{3x^4+7}$?

Ex. 8. What is the integral of $\frac{x^2-2x}{2x^3-6x^2+1}dx$?

Ex. 9. What is the integral of $\frac{3x^2dx}{x^3+a^2}$?

Ex. 10. What is the integral of $\frac{3x^2+2x+1}{x^3+x^2+x+1}dx$?

Ex. 11. What is the integral of $\frac{xdx}{a+bx}$?

Ans. $\frac{x}{b}-\frac{a}{b^2}\log.(a+bx)$.

Ex. 12. What is the integral of $\frac{x^5dx}{x^2+a^2}$?

NOTE.—Divide the numerator by the denominator.

Ans. $\frac{x^4}{4}-\frac{a^2x^2}{2}+\frac{a^4}{2}\log.(x^2+a^2)$.

160. *Certain polynomial differentials.* Every polynomial of the form $(a+bx+cx^2+\text{etc.})^m x^n dx$, in which m is a positive whole number, may be integrated *by raising the quantity within the parenthesis to the mth power, multiplying each term by $x^n dx$, and then integrating each term separately.*

Ex. 1. What is the integral of $(a+bx)^2dx$?

Ans. $a^2x+abx^2+\frac{b^2x^3}{3}+\text{C}$.

Ex. 2. What is the integral of $(5+7x^2)^2dx$?

Ans. $25x+\frac{70x^3}{3}+\frac{49x^5}{5}+\text{C}$.

Ex. 3. What is the integral of $(a+3x^2)^3dx$?

Ex. 4. What is the integral of $(a+bx^2)^2x^3dx$?

Ex. 5. What is the integral of $(a-x^2)^3x^{\frac{1}{2}}dx$?

Ex. 6. What is the integral of $(a-bx^{\frac{1}{2}})^2x^{-\frac{1}{3}}dx$?

Ex. 7. What is the integral of $(a+bx^3)^2x^2dx$?

Ex. 8. What is the integral of $(a+bx^3)^3x^4dx$?

Ex. 9. What is the integral of $(a^2+bx^4)^3x^5dx$?

161. *Particular binomial differentials.* By the introduction of an auxiliary variable, many binomial expressions may be transformed into monomials, and then integrated as in Art. 157.

Let $du=(a+bx^n)^m x^{n-1}dx$,

in which the exponent of the variable without the parenthesis is one less than the exponent of the variable within.

Let us put $$y=a+bx^n.$$

Then $$dy=bnx^{n-1}dx,$$

and $$x^{n-1}dx=\frac{dy}{bn}.$$

Hence $$du=\frac{y^m dy}{bn};$$

and integrating, we have

$$u=\frac{y^{m+1}}{(m+1)bn}+\mathrm{C};$$

and replacing the value of y, we have

$$u=\frac{(a+bx^n)^{m+1}}{(m+1)bn}+\mathrm{C};$$

that is, to integrate a binomial differential when the exponent of the variable without the parenthesis is one less than that within, *increase the primitive exponent of the binomial by unity, and divide the result by the product of the new exponent, the coefficient, and the exponent of the variable within the parenthesis.*

Ex. 1. What is the integral of $(a+3x^2)^3xdx$?

$$Ans.\ \frac{(a+3x^2)^4}{4.3.2}+\mathrm{C}=\frac{(a+3x^2)^4}{24}+\mathrm{C}.$$

Ex. 2. What is the integral of $(a+bx^2)^{\frac{1}{2}}mxdx$?

$$Ans.\ \frac{m}{3b}(a+bx^2)^{\frac{3}{2}}+\mathrm{C}.$$

Ex. 3. What is the integral of $\frac{xdx}{\sqrt{a^2+x^2}}$?

$$Ans.\ \sqrt{a^2+x^2}+\mathrm{C}.$$

Ex. 4. What is the integral of $(a+bx^2)^{\frac{3}{2}}exdx$?

$$Ans.\ \frac{e(a+bx^2)^{\frac{5}{2}}}{5b}+\mathrm{C}.$$

Ex. 5. What is the integral of $(a^2+x^n)^3anx^{n-1}dx$?

Ex. 6. What is the integral of $(3-4x^5)^{-\frac{1}{2}}2x^4dx$?

$$Ans.\ -\frac{(3-4x^5)^{\frac{1}{2}}}{5}+\mathrm{C}.$$

162. *Binomial differentials.* Every binomial differential of the form $du = A(a+bx)^m x^n dx$ can be integrated when either m or n is a positive whole number, and the exponent of x in the parenthesis is unity.

If m be a whole number and positive, we may integrate as in Art. 158.

If n be a whole number and positive, m being either fractional or negative, place

$$y = a+bx;$$

then

$$x = \frac{y-a}{b},$$

$$dx = \frac{dy}{b}, \quad du = A\left(\frac{y-a}{b}\right)^n y^m \frac{dy}{b},$$

$$u = \frac{A}{b}\int \left(\frac{y-a}{b}\right)^n y^m dy,$$

which may be integrated as in Art. 158. If we replace the value of y, the integral will be expressed in terms of x.

Ex. 1. What is the integral of $x^2(a+x)^{\frac{1}{2}}dx$?

Put $y = a+x$; then $x = y-a$, $dx = dy$,

$$u = \int (y-a)^2 y^{\frac{1}{2}} dy = \frac{2}{7}y^{\frac{7}{2}} - \frac{4}{5}ay^{\frac{5}{2}} + \frac{2}{3}a^2 y^{\frac{3}{2}},$$

$$= \frac{2}{35}y^{\frac{3}{2}}(5y^2 - 14ay + \frac{35}{3}a^2);$$

or, replacing the value of y,

$$u = \frac{2}{35}(a+x)^{\frac{3}{2}}(5x^2 - 4ax + \frac{8}{3}a^2) + C.$$

Ex. 2. What is the integral of $\dfrac{xdx}{(a+bx)^{\frac{1}{2}}}$?

Place the expression under the form

$$du = x(a+bx)^{-\frac{1}{2}}dx; \text{ whence } du = \frac{y-a}{b^2}y^{-\frac{1}{2}}dy.$$

Hence $$u = \frac{y^{\frac{1}{2}}}{b^2}\left(\frac{2}{3}y - 2a\right) = \frac{2}{3b^2}(a+bx)^{\frac{1}{2}}(bx-2a) + C.$$

Ex. 3. What is the integral of $\frac{x^2dx}{(a+bx)^{\frac{1}{2}}}$?

$$Ans.\ \frac{2}{b^3}(a+bx)^{\frac{1}{2}}\left\{\frac{(a+bx)^2}{5}-\frac{2a}{3}(a+bx)+a^2\right\}+C.$$

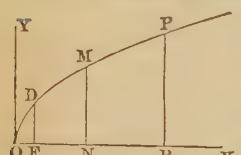
Ex. 4. What is the integral of $\frac{x^3dx}{(1+x)^{\frac{1}{2}}}$?

$$Ans.\ \frac{2}{7}(1+x)^{\frac{7}{2}}-\frac{6}{5}(1+x)^{\frac{5}{2}}+2x(1+x)^{\frac{1}{2}}+C.$$

Ex. 5. What is the integral of $\frac{xdx}{(a+bx)^{\frac{1}{3}}}$?

$$Ans.\ \frac{3}{b^2}(a+bx)^{\frac{2}{3}}\left(\frac{a+bx}{5}-\frac{a}{2}\right)+C.$$

163. *Geometrical illustration.* The problem of finding the areas of curves was one of those which gave rise to the integral calculus, and affords an illustration of the preceding principles. Let OMP be a parabola whose equation is $y^2=mx$, and suppose it is required to find the area included between this curve and the axis of X. Let A represent the required area. Then (Art. 122) $dA=ydx$. But for the parabola $dx=\frac{2ydy}{m}$; hence



$$dA=ydx=\frac{2y^2dy}{m}.$$

Therefore $A=\frac{2y^3}{3m}+C=\frac{2}{3m}(mx)^{\frac{3}{2}}+C=\frac{2}{3}m^{\frac{1}{2}}x^{\frac{3}{2}}+C,$ (1)

which is a general expression for the required area.

164. *Definite integral.* To complete each integral as determined by the preceding rules, we have added a constant quantity C. While the value of this constant is unknown, the expression is called an *indefinite integral;* but in the application of the calculus to the solution of particular problems, the value

of the constant C may be determined from the conditions of the problem. When there is any thing in the nature of the problem from which we can know the value of the function for some particular value of the variable, by substituting this value in the indefinite integral the value of the constant C can be found; and since C is constant, if we find its value for a particular value of the variable, we know its value for *all* values of the variable.

When the value of the constant has been determined, and a particular value assigned to the independent variable, the value of the integral is known, and it is called a *definite integral.*

To determine the value of C in the example of Art. 163, we perceive that if we estimate the area from O, the integral which expresses the required area becomes 0 when $x=0$. Hence equation (1) becomes

$$0=0+\mathrm{C}.$$

Therefore the constant C is equal to 0, and the entire integral is

$$\mathrm{A}=\frac{2y^3}{3m}=\frac{2}{3}m^{\frac{1}{2}}x^{\frac{3}{2}}.$$

This represents the area estimated from O to any value we may choose to give to x. If $x=\mathrm{OR}$, A = area OPR.

If, however, we estimate the area from the ordinate DF, which passes through the focus F, calling $\mathrm{A}=0$ when $x=\frac{m}{4}$, we have

$$0=\frac{2}{3}m^{\frac{1}{2}}\left(\frac{m}{4}\right)^{\frac{3}{2}}+\mathrm{C}=\frac{m^2}{12}+\mathrm{C}.$$

Hence $\mathrm{C}=-\frac{m^2}{12}$, and the corrected integral is

$$\mathrm{A}=\frac{2}{3}m^{\frac{1}{2}}x^{\frac{3}{2}}-\frac{m^2}{12}.$$

This represents the area estimated from the focal ordinate DF to any value we may choose to give to x. If $x=\mathrm{OR}$, this corrected integral represents the area FDPR.

165. *Integral between limits.* Suppose we wish to obtain an expression for the curvilinear area MNRP, contained be-

tween the two ordinates MN, PR. We first obtain the area of the portion OMN. Suppose the variable x to be equal to ON, which we will represent by a; then the area of OMN will be expressed by

$$\frac{2}{3}m^{\frac{1}{2}}a^{\frac{3}{2}}+\mathrm{C}.$$

Next suppose the variable x to become equal to OR, which we will represent by b; then the area of OPR will be expressed by

$$\frac{2}{3}m^{\frac{1}{2}}b^{\frac{3}{2}}+\mathrm{C}.$$

Subtracting the former expression from the latter, we obtain the value of the curvilinear area MNRP,

$$\frac{2}{3}m^{\frac{1}{2}}b^{\frac{3}{2}}-\frac{2}{3}m^{\frac{1}{2}}a^{\frac{3}{2}}=\frac{2m^{\frac{1}{2}}}{3}(b^{\frac{3}{2}}-a^{\frac{3}{2}}).$$

Hence we find that the constant C may be made to disappear by giving two successive values to the independent variable, and taking the difference between the two integrals corresponding to these values.

When we take the excess of the value of an integral when the independent variable has become equal to b, above the value when it was only equal to a, we are said to integrate between the limits of $x=a$ and $x=b$.

The integration between the limits a and b is indicated by the sign $\int_a^b$.

Thus $$\int_a^b m^{\frac{1}{2}}x^{\frac{1}{2}}dx=\frac{2m^{\frac{1}{2}}}{3}(b^{\frac{3}{2}}-a^{\frac{3}{2}}).$$

Ex. 1. Integrate $\int_a^b 2x\,dx$, and illustrate the case by a geometrical example. *Ans.*

Ex. 2. Integrate $\int_a^b 3x^2dx$; illustrate the case by a geometrical example, and determine the numerical value when $a=4$ and $b=6$. *Ans.*

Ex. 3. Integrate $\int_a^b \frac{\pi}{2} x dx$; illustrate the case by a geometrical example, and determine the value of the definite integral when $a=2$ and $b=3$. *Ans.*

Ex. 4. Integrate $\int_a^b \frac{\pi}{2} x^2 dx$; illustrate the case by a geometrical example, and determine the value of the definite integral when $a=4$ and $b=6$. *Ans.*

Ex. 5. What is the value of $\int_a^b 2(e+x)dx$, when $a=10, b=20$, and $e=4$? and illustrate the example by a geometrical figure. *Ans.*

Ex. 6. What is the value of $\int_a^b 3(e+nx^2)^2 2nx dx$, when $a=4$, $b=6$, $e=4$, and $n=2$? *Ans.*

Ex. 7. What is the value of $\int_a^b \frac{dx}{e+x}$, when $a=2$, $b=3$, and $e=4$? *Ans.*

INTEGRATION OF THE DIFFERENTIALS OF TRIGONOMETRICAL FUNCTIONS.

166. *Integration of elementary trigonometrical expressions.*

(1) We have found, in Art. 44, that if x represents an arc of a circle, $d \sin. x = \cos. x dx$.

Hence $\int \cos. x dx = \sin. x + C.$

(2) We have found, in Art. 45, that $d \cos. x = -\sin. x dx$.

Hence $\int \sin. x dx = -\cos. x + C.$

(3) We have found, in Art. 45, that $d \text{ ver. sin. } x = \sin. x dx$.

Hence $\int \sin. x dx = \text{ver. sin. } x + C.$

(4) We have found, in Art. 46, that $d \text{ tang. } x = \sec.^2 x dx$, or $\frac{dx}{\cos.^2 x}$.

Hence $\int \sec.^2 x dx$, or $\int \frac{dx}{\cos.^2 x} = \text{tang. } x + C.$

(5) We have found, in Art. 47, that $d\,\text{cotang.}\,x = -\text{cosec.}^2 x dx$, or $-\dfrac{dx}{\text{sin.}^2 x}$.

Hence $$\int \frac{dx}{\text{sin.}^2 x} = -\text{cotang.}\,x + \text{C}.$$

(6) We have found, in Art. 48, that $d\,\text{sec.}\,x = \text{tang.}\,x\,\text{sec.}\,x dx$.

Hence $$\int \text{tang.}\,x\,\text{sec.}\,x dx = \text{sec.}\,x + \text{C}.$$

(7) We have found, in Art. 49, that $d\,\text{cosec.}\,x = -\text{cotang.}\,x\,\text{cosec.}\,x dx$.

Hence $$\int \text{cotang.}\,x\,\text{cosec.}\,x dx = -\text{cosec.}\,x + \text{C}.$$

167. *Integration of more complicated expressions.* Expressions which do not appear to be of the form for the application of the previous methods may sometimes be brought to the required form by reduction.

Ex. 1. Find the integral of $\text{cos.}\,nxdx$.

$$\int \text{cos.}\,nxdx = \frac{1}{n}\int \text{cos.}\,nxd(nx) = \frac{1}{n}\,\text{sin.}\,nx + \text{C}.$$

Ex. 2. Find the integral of $\text{sin.}\,(m+n)xdx$.

$$Ans.\ -\frac{\text{cos.}\,(m+n)x}{m+n} + \text{C}.$$

Ex. 3. Find the integral of $\text{tang.}\,xdx$.

$$\int \text{tang.}\,xdx = \int \frac{\text{sin.}\,xdx}{\text{cos.}\,x} = \int \frac{-d\,\text{cos.}\,x}{\text{cos.}\,x} = -\text{log.}\,\text{cos.}\,x + \text{C}.$$

Ex. 4. Find the integral of $\text{cotang.}\,xdx$.

$$Ans.\ \text{log.}\,\text{sin.}\,x + \text{C}.$$

Ex. 5. Find the integral of $\dfrac{dx}{1+\text{cos.}\,x}$.

By Trig., Art. 74,

$$1 + \text{cos.}\,x = 2(\text{cos.}\,\tfrac{1}{2}x)^2.$$

Hence

$$\int \frac{dx}{1+\text{cos.}\,x} = \int \frac{dx}{2(\text{cos.}\,\frac{1}{2}x)^2} = \int \frac{d(\frac{1}{2}x)}{(\text{cos.}\,\frac{1}{2}x)^2} = \text{tang.}\,\tfrac{1}{2}x + \text{C},$$

by Art. 166, (4).

Ex. 6. Find the integral of $\frac{\sin. x dx}{1+\cos. x}$.

$$\int \frac{\sin. x dx}{1+\cos. x} = \int \frac{-d(1+\cos. x)}{1+\cos. x} = -\log. (1+\cos. x) + C.$$

Ex. 7. Find the integral of $\frac{dx}{\sin. x \cos. x}$.

$$\int \frac{dx}{\sin. x \cos. x} = \int \frac{\sec.^2 x dx}{\tang. x} = \int \frac{d \tang. x}{\tang. x} = \log. \tang. x + C.$$

Ex. 8. Find the integral of $\frac{dx}{\sin. x}$.

By Trig., Art. 74,

$$\sin. x = 2 \sin. \tfrac{1}{2}x \cos. \tfrac{1}{2}x = \frac{2 \tang. \frac{1}{2}x}{(\sec. \frac{1}{2}x)^2}.$$

Hence $$\frac{dx}{\sin. x} = \frac{(\sec. \frac{1}{2}x)^2 dx}{2 \tang. \frac{1}{2}x} = \frac{(\sec. \frac{1}{2}x)^2 d(\frac{1}{2}x)}{\tang. \frac{1}{2}x} = \frac{d \tang. \frac{1}{2}x}{\tang. \frac{1}{2}x}.$$

Therefore $$\int \frac{dx}{\sin. x} = \log. \tang. \tfrac{1}{2}x + C.$$

Ex. 9. Find the integral of $\tang.^2 x dx$.

$$\int \tang.^2 x dx = \int (\sec.^2 x - 1) dx = \tang. x - x + C.$$

Ex. 10. Find the integral of $\sin.^2 x dx$.

By Trig., Art. 74, $\sin.^2 x = \frac{1}{2} - \frac{1}{2} \cos. 2x$.

Hence $$\sin.^2 x dx = \frac{dx}{2} - \frac{1}{2} \cos. 2x dx.$$

Therefore $$\int \sin.^2 x dx = \tfrac{1}{2}x - \frac{1}{4} \int \cos. 2x d(2x) = \frac{x}{2} - \frac{1}{4} \sin. 2x + C.$$

168. *Differential of arc in terms of sine.* We have found (Art. 53) that if y designates an arc, and x its sine, the radius of the circle being unity,

$$dy = \frac{dx}{\sqrt{1-x^2}}.$$

Hence $$\int \frac{dx}{\sqrt{1-x^2}} = y + C = \sin.^{-1} x + C.$$

If the arc be estimated from the beginning of the first quadrant, the sine will be 0 when the arc is 0, and consequently C equals zero.

Therefore the complete integral is

$$\int \frac{dx}{\sqrt{1-x^2}} = \text{the arc of which } x \text{ is the sine.}$$

If it be required to integrate an expression of the form

$$dy = \frac{dx}{\sqrt{a^2-x^2}}, \qquad (1)$$

it may be done by the aid of an auxiliary variable.

Assume $v = \frac{x}{a}$, or $x = av$;

then $dx = a\,dv$, and $\sqrt{a^2-x^2} = a\sqrt{1-v^2}$.

Substituting these values in equation (1), we have

$$dy = \frac{dv}{\sqrt{1-v^2}}.$$

Hence y = the arc whose sine is v = the arc whose sine is $\frac{x}{a}$; that is,

$$\int \frac{dx}{\sqrt{a^2-x^2}} = \sin.^{-1}\frac{x}{a} + C.$$

Ex. 1. Integrate the expression $dy = \frac{dx}{\sqrt{4-x^2}}$.

Ans. y = arc whose sine is $\frac{x}{2} + C$.

Ex. 2. Integrate the expression $\frac{5dx}{\sqrt{3-x^2}}$.

Ans. $5 \sin.^{-1}\frac{x}{\sqrt{3}} + C$.

Ex. 3. Integrate the expression $\frac{3dx}{\sqrt{3-2x^2}}$.

Ans. $\frac{3}{\sqrt{2}} \sin.^{-1} x\sqrt{\frac{2}{3}} + C$.

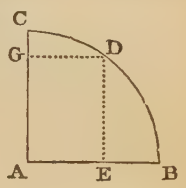
169. *Differential of arc in terms of cosine.* We have found,

in Art. 53, that if y designates an arc, and x its cosine, the radius of the circle being unity,

$$dy=\frac{-dx}{\sqrt{1-x^2}}.$$

Hence $$\int\frac{-dx}{\sqrt{1-x^2}}=y+\mathrm{C}=\cos.^{-1}x+\mathrm{C}. \qquad (1)$$

To determine the constant C, we observe that if the arc be estimated from B, the beginning of the first quadrant, the cosine becomes 0 when the arc becomes a quadrant, which is represented by $\frac{\pi}{2}$. Making, therefore, $\int\frac{-dx}{\sqrt{1-x^2}}=\frac{\pi}{2}$, and $x=0$ in equation (1), we have



$$\frac{\pi}{2}=\text{arc whose cosine is } 0+\mathrm{C}; \qquad (2)$$

and since the arc whose cosine is 0 is equal to $\frac{\pi}{2}$, Eq. 2 becomes

$$\frac{\pi}{2}=\frac{\pi}{2}+\mathrm{C}.$$

Hence C = 0, and the complete integral is

$$\int\frac{-dx}{\sqrt{1-x^2}}=\cos.^{-1}x.$$

If it be required to integrate an expression of the form

$$\frac{-dx}{\sqrt{a^2-x^2}},$$

it may be done by the aid of an auxiliary variable, as in Art. 168, and we shall find

$$\int\frac{-dx}{\sqrt{a^2-x^2}}=\cos.^{-1}\frac{x}{a}.$$

Ex. 1. Integrate the expression $\frac{-2dx}{\sqrt{4-x^2}}$.

Ans. $2\cos.^{-1}\frac{x}{2}+\mathrm{C}.$

Ex. 2. Integrate the expression $\frac{-dx}{\sqrt{9-4x^2}}$.

Ans. $\frac{1}{2}\cos.^{-1}\frac{2x}{3}+\mathrm{C}.$

170. *Differential of arc in terms of versed sine.* We have found, Art. 53, that if y represents an arc, and x its versed sine,

$$dy=\frac{dx}{\sqrt{2x-x^2}}.$$

Hence $$\int\frac{dx}{\sqrt{2x-x^2}}=\text{ver. sin.}^{-1}x+\text{C}.$$

If the arc be estimated from the beginning of the first quadrant, we shall have $\text{C}=0$, and the complete integral is

$$\int\frac{dx}{\sqrt{2x-x^2}}=\text{the arc of which } x \text{ is the versed sine.}$$

If it be required to integrate an expression of the form

$$dy=\frac{dx}{\sqrt{2ax-x^2}}, \qquad (1)$$

it may be done by the aid of an auxiliary variable.

Assume $$v=\frac{x}{a}, \text{ or } x=av;$$

then $$dx=adv.$$

Substituting in equation (1), we have

$$dy=\frac{adv}{\sqrt{2a^2v-a^2v^2}}=\frac{dv}{\sqrt{2v-v^2}}.$$

Hence $y=$the arc whose versed sine is v, or $\frac{x}{a}$; that is,

$$\int\frac{dx}{\sqrt{2ax-x^2}}=\text{ver. sin.}^{-1}\frac{x}{a}.$$

Ex. 1. Integrate the expression $\frac{5dx}{\sqrt{4x-2x^2}}$.

Ans. $\frac{5}{\sqrt{2}}\text{ ver. sin.}^{-1}x.$

Ex. 2. Integrate the expression $\frac{dx}{\sqrt{6x-9x^2}}$.

Ans. $\frac{1}{3}\text{ ver. sin.}^{-1}3x.$

Ex. 3. Integrate the expression $\frac{3dx}{\sqrt{9x-4x^2}}$.

Ans. $\frac{3}{2}\text{ ver. sin.}^{-1}\frac{8x}{9}.$

171. *Differential of arc in terms of tangent.* We have found, in Art. 53, that if y designates an arc, and x its tangent,

$$dy=\frac{dx}{1+x^2}.$$

Hence $$\int\frac{dx}{1+x^2}=\text{tang.}^{-1}x+\text{C}.$$

If the arc be estimated from the beginning of the first quadrant, we shall have $y=0$ when $\int\frac{dx}{1+x^2}=0$; hence $\text{C}=0$ and the complete integral is

$$\int\frac{dx}{1+x^2}=\text{the arc of which } x \text{ is the tangent.}$$

If it be required to integrate an expression of the form

$$dy=\frac{dx}{a^2+x^2}, \qquad (1)$$

it may be done by the aid of an auxiliary variable.

Assume $$v=\frac{x}{a}, \text{ or } x=av;$$

then $$dx=adv.$$

Substituting in equation (1), we have

$$dy=\frac{adv}{a^2+a^2v^2}=\frac{1}{a}\times\frac{dv}{1+v^2}.$$

Hence $y=\frac{1}{a}\times$ arc whose tangent is v, or $\frac{x}{a}$; that is,

$$\int\frac{dx}{a^2+x^2}=\frac{1}{a}\,\text{tang.}^{-1}\frac{x}{a}.$$

Ex. 1. Integrate the expression $\frac{dx}{2+3x^2}$.

Ans.

$$\int\frac{dx}{2+3x^2}=\frac{1}{3}\int\frac{dx}{\frac{2}{3}+x^2}=\frac{1}{3}\times\frac{1}{\sqrt{\frac{2}{3}}}\,\text{tang.}^{-1}\frac{x}{\sqrt{\frac{2}{3}}}=\frac{1}{\sqrt{6}}\,\text{tang.}^{-1}x\sqrt{\tfrac{3}{2}}.$$

Ex. 2. Integrate the expression $\frac{3dx}{4+9x^2}$. *Ans.* $\frac{1}{2}\,\text{tang.}^{-1}\frac{3x}{2}$.

Ex. 3. Integrate the expression $\frac{dx}{1+5x^2}$.

Ans. $\frac{1}{\sqrt{5}}$ tang.$^{-1}x\sqrt{5}$.

172. *Differential of arc in terms of secant.* We have found, Art. 53, that if y represents an arc, and x its secant,

$$dy=\frac{dx}{x\sqrt{x^2-1}}.$$

Hence $$\int\frac{dx}{x\sqrt{x^2-1}}=\text{sec.}^{-1}x+\text{C}.$$

If the arc be estimated from the beginning of the first quadrant, C$=0$, and the complete integral is

$$\int\frac{dx}{x\sqrt{x^2-1}}=\text{sec.}^{-1}x.$$

If it were required to integrate an expression of the form

$$dy=\frac{dx}{x\sqrt{x^2-a^2}},$$

we may proceed as in Art. 168, and we shall find

$$\int\frac{dx}{x\sqrt{x^2-a^2}}=\frac{1}{a}\text{sec.}^{-1}\frac{x}{a}.$$

Ex. 1. Integrate the expression $\frac{dx}{x\sqrt{2x^2-3}}$.

Ans. $\frac{1}{\sqrt{3}}$ sec.$^{-1}x\sqrt{\frac{2}{3}}$.

Ex. 2. Integrate the expression $\frac{2dx}{x\sqrt{3x^2-4}}$.

Ans. sec.$^{-1}x\frac{\sqrt{3}}{2}$.

CHAPTER II.

RATIONAL FRACTIONS. INTEGRATION OF CERTAIN IRRATIONAL DIFFERENTIALS.

173. By the term *rational fraction* is to be understood a fraction in which the exponents of the variable are all positive integers. If the numerator contain powers of x as high or higher than any in the denominator, we may, by division, reduce the fraction to the form of an entire and rational part, plus a fraction in which the numerator is of a lower degree than the denominator. The former can be integrated by methods already explained, and therefore we need only consider those rational fractions in which the highest exponent of the variable in the numerator is at least one less than the highest exponent in the denominator. In order to integrate such a fraction, we resolve it into the sum of a number of fractions with simpler denominators, called *partial fractions.* This can always be done by the method of undetermined coefficients explained in Algebra, Art. 375. There may be four cases:

I. When the denominator of the given fraction can be resolved into n real and unequal factors of the first degree.

II. When the denominator can be resolved into n real factors of the first degree, and some of them are equal.

III. When some of the binomial factors are imaginary, but no two are equal.

IV. When some of the binomial factors are imaginary and equal.

174. Case I. *When the denominator can be resolved into n real and unequal factors of the first degree.*

Ex. 1. Integrate the expression $\frac{dx}{x^2-2x}$, or $\frac{dx}{x(x-2)}$.

The fraction $\frac{1}{x(x-2)}$ can be decomposed into $\frac{1}{2}\left\{\frac{1}{x-2}-\frac{1}{x}\right\}$.

Hence $$\int\frac{dx}{x(x-2)}=\frac{1}{2}\int\left\{\frac{dx}{x-2}-\frac{dx}{x}\right\}$$
$$=\tfrac{1}{2}\{\log.(x-2)-\log.x\}=\tfrac{1}{2}\log.\frac{x-2}{x}.$$

Ex. 2. Integrate the expression $\frac{adx}{x^2-ax}$. *Ans.* $\log.\frac{x-a}{x}$.

Ex. 3. Integrate the expression $\frac{xdx}{x^2+6x+8}$, or $\frac{xdx}{(x+4)(x+2)}$.

Assume $$\frac{x}{(x+4)(x+2)}=\frac{A}{x+4}+\frac{B}{x+2}. \qquad (1)$$

Clearing of fractions, we have

$$x=Ax+2A+Bx+4B.$$

Since this equation is true for all values of x, by the principle of undetermined coefficients we may put the coefficients of the like powers of x in the two members equal to each other, and we have $1=A+B$, and $0=2A+4B$;

whence $A=+2$, and $B=-1$.

Substituting these values in equation (1), we have

$$\frac{x}{(x+4)(x+2)}=\frac{2}{x+4}-\frac{1}{x+2}.$$

Hence $$\int\frac{xdx}{(x+4)(x+2)}=\int\frac{2dx}{x+4}-\int\frac{dx}{x+2}$$
$$=2\log.(x+4)-\log.(x+2)=\log.\frac{(x+4)^2}{x+2}.$$

If there are n factors in the denominator of the given expression, there will be n corresponding fractions of the form $\frac{A}{x+a}$, $\frac{B}{x+b}$, etc.; and when these are reduced to a common denominator with the first member, they will give an identical equation in which the coefficients of the like powers of x in the two members are equal, and hence the values of A, B, etc., may be determined. Hence, to decompose the fraction into partial fractions and integrate it, we have the following rule:

Assume the primitive fraction (omitting the factor dx)

equal to the sum of as many partial fractions of the form $\frac{A}{x+a}$ *as there are factors of the first degree in its denominator, the denominators being the several factors of the given denominator. Clear the equation of denominators; equate the coefficients of the like powers of* x, *and thence determine the values of* A, B, *etc. Substitute these values of* A, B, *etc., in the assumed fractions, multiply by* dx, *and take the sum of the integrals of the partial fractions.*

Ex. 4. Integrate the expression $\frac{a\,dx}{x^2-5x+6}$.

Ans. $a \log. \frac{x-3}{x-2}+C.$

Ex. 5. Integrate the expression $\frac{(1+3x^2)dx}{x-x^3}$.

The factors of the denominator are $1+x$, $1-x$, and x.

Hence we assume $\frac{1+3x^2}{x-x^3}=\frac{A}{1+x}+\frac{B}{1-x}+\frac{C}{x}$.

Clearing of denominators, we have

$$1+3x^2=Ax-Ax^2+Bx+Bx^2+C-Cx^2; \quad (1)$$

whence $3=-A+B-C;\ 0=A+B;\ 1=C.$

Therefore $A=-2$, and $B=2$.

Hence $\int\frac{(1+3x^2)dx}{x-x^3}=-2\int\frac{dx}{1+x}+2\int\frac{dx}{1-x}+\int\frac{dx}{x}$

$$=-2 \log. (1+x)-2 \log. (1-x)+\log. x$$

$$=-\log. (1-x^2)^2+\log. x=\log. \frac{x}{(1-x^2)^2}+C.$$

In some cases it will facilitate the finding of the values of A, B, C, etc., if we merely indicate the factors in clearing the equation of fractions; and then, since equation (1) holds true for every value of x, give to x such values as to render the factors in turn equal to zero.

Thus in the equation above we have

$$1+3x^2=Ax(1-x)+Bx(1+x)+C(1+x)(1-x).$$

Making $x=0$, we have $1=C$;
" $x=1$, " $4=2B$;
" $x=-1$, " $4=-2A$;

the same values as above.

Ex. 6. Integrate the expression $\frac{(a^3+bx^2)dx}{a^2x-x^3}$.

$$Ans.\ a\log.x-\frac{a+b}{2}\log.(a^2-x^2)+C.$$

Ex. 7. Integrate the expression $\frac{(3x-5)dx}{x^2-6x+8}$.

$$Ans.\ \frac{7}{2}\log.(x-4)-\frac{1}{2}\log.(x-2)=\log.\frac{(x-4)^{\frac{7}{2}}}{(x-2)^{\frac{1}{2}}}.$$

Ex. 8. Integrate the expression $\frac{x^{-1}dx}{a+bx}$. $Ans.\ \frac{1}{a}\log.\frac{x}{a+bx}$.

Ex. 9. Integrate the expression $\frac{(2x+3)dx}{x^3+x^2-2x}$.

$$Ans.\ \frac{5}{3}\log.(x-1)-\frac{1}{6}\log.(x+2)-\frac{3}{2}\log.x.$$

175. Case II. *When some of the binomial factors of the denominator are real and equal.* If all the factors of the denominator are equal, the denominator will be of the form $(x-a)^n$, and the fraction may be integrated as in Art. 158. It is only necessary, then, to consider the case in which a portion of the factors are equal. It will be found by trial that the method of the preceding article is not applicable in this case. Thus, take the example $\frac{x^2+x}{(x-2)^2(x-1)}$, in which two of the factors are equal to $x-2$. According to the method of Art. 174, we should have

$$\frac{x^2+x}{(x-2)^2(x-1)}=\frac{A}{x-2}+\frac{B}{x-2}+\frac{C}{x-1}=\frac{A+B}{x-2}+\frac{C}{x-1},$$

where $A+B$ must be regarded as a single constant. Reducing the two members to a common denominator, and placing the coefficients of the like powers of x in the two members equal

to each other, we shall have three independent equations containing but two unknown quantities.

We may obviate this difficulty by assuming the given fraction equal to $$\frac{A}{(x-2)^2}+\frac{B}{x-2}+\frac{C}{x-1}.$$

Clearing of denominators, we have

$$x^2+x=Ax-A+Bx^2-3Bx+2B+Cx^2-4Cx+4C;$$

whence $1=B+C$, $1=A-3B-4C$, $0=-A+2B+4C$.

Therefore $A=6$, $B=-1$, and $C=2$,

and $$\frac{x^2+x}{(x-2)^2(x-1)}=\frac{6}{(x-2)^2}-\frac{1}{x-2}+\frac{2}{x-1}.$$

Hence $$\int\frac{(x^2+x)dx}{(x-2)^2(x-1)}=\int\frac{6dx}{(x-2)^2}-\int\frac{dx}{x-2}+\int\frac{2dx}{x-1}$$

$$=-\frac{6}{x-2}-\log.(x-2)+2\log.(x-1)$$

$$=-\frac{6}{x-2}+\log.\frac{(x-1)^2}{x-2}.$$

If the denominator of the given fraction contain n equal factors of the first degree, we must assume n partial fractions of the form

$$\frac{A}{(x-a)^n}+\frac{B}{(x-a)^{n-1}}+\frac{C}{(x-a)^{n-2}}+\ \ldots\ +\frac{N}{x-a}.$$

Having determined A, B, C, etc., each partial fraction is to be multiplied by dx, and may be integrated as in Art. 174.

Ex. 2. Integrate the expression $\dfrac{xdx}{(x+2)^2(x+1)}$.

$$Ans.\ -\frac{2}{x+2}+\log.\frac{x+2}{x+1}.$$

Ex. 3. Integrate the expression $\dfrac{dx}{(x+2)^2(x-1)}$.

$$Ans.\ \frac{1}{3(x+2)}+\tfrac{1}{9}\log.\frac{x-1}{x+2}.$$

Ex. 4. Integrate the expression $\dfrac{(x^2-2)dx}{x^4-x^3}$, or $\dfrac{(x^2-2)dx}{x^3(x-1)}$.

Assume $$\frac{x^2-2}{x^3(x-1)}=\frac{A}{x^3}+\frac{B}{x^2}+\frac{C}{x}+\frac{D}{x-1};$$

whence $x^2-2=Ax-A+Bx^2-Bx+Cx^3-Cx^2+Dx^3;$

$$0=C+D,\ 1=B-C,\ 0=A-B,\ -2=-A;$$

$$A=2,\ B=2,\ C=1,\ D=-1.$$

Hence $$\frac{x^2-2}{x^3(x-1)}=\frac{2}{x^3}+\frac{2}{x^2}+\frac{1}{x}-\frac{1}{x-1};$$

$$\int\frac{(x^2-2)dx}{x^3(x-1)}=\int\frac{2dx}{x^3}+\int\frac{2dx}{x^2}+\int\frac{dx}{x}-\int\frac{dx}{x-1}$$

$$=-\frac{1}{x^2}-\frac{2}{x}+\log.x-\log.(x-1)$$

$$=-\frac{1+2x}{x^2}+\log.\frac{x}{x-1}+C.$$

Ex. 5. Integrate the expression $\frac{(x+10)dx}{x^3-3x^2}$.

$$Ans.\ \frac{10}{3x}+\log.\left(\frac{x-3}{x}\right)^{\frac{13}{9}}.$$

Ex. 6. Integrate the expression $\frac{xdx}{(a+bx)^3}$.

$$Ans.\ -\frac{a+2bx}{2b^2(a+bx)^2}.$$

176. Case III. *When some of the binomial factors of the denominator are imaginary but unequal.* It is shown in Algebra, Art. 443, that imaginary roots are found only in pairs; hence the product of each pair of imaginary factors will be of the form $(x-a)^2+b^2$. Instead of assuming a partial fraction for each imaginary factor, we assume for each pair of imaginary factors a partial fraction of the form

$$\frac{Ax+B}{(x-a)^2+b^2}.$$

By clearing of denominators, and proceeding as in the last Article, the values of A and B may be determined.

Ex. 1. Integrate the expression $\frac{x^2dx}{x^3+x^2+x+1}$, or $\frac{x^2dx}{(x^2+1)(x+1)}$, where x^2+1 is a quadratic factor whose roots are imaginary.

Assume $$\frac{x^2}{(x^2+1)(x+1)}=\frac{Ax+B}{x^2+1}+\frac{C}{x+1}.$$

Clearing of fractions, we have

$$x^2=Ax^2+Ax+Bx+B+Cx^2+C;$$

whence $$1=A+C,\ 0=A+B,\ 0=B+C;$$

$$A=\tfrac{1}{2},\ B=-\tfrac{1}{2},\ C=\tfrac{1}{2};$$

$$\frac{x^2}{(x^2+1)(x+1)}=\frac{x-1}{2(x^2+1)}+\frac{1}{2(x+1)};$$

$$\int\frac{x^2dx}{x^3+x^2+x+1}=\tfrac{1}{2}\int\frac{xdx}{x^2+1}-\tfrac{1}{2}\int\frac{dx}{x^2+1}+\tfrac{1}{2}\int\frac{dx}{x+1}$$

$$=\tfrac{1}{4}\log.(x^2+1)-\tfrac{1}{2}\tan g.^{-1}x+\tfrac{1}{2}\log.(x+1).$$

Ex. 2. Integrate the expression $\frac{dx}{x^3-x^2+x-1}$.

$$Ans.\ -\tfrac{1}{2}\log\frac{\sqrt{x^2+1}}{x-1}-\tfrac{1}{2}\tan g.^{-1}x.$$

Ex. 3. Integrate the expression $\frac{(x^2-x+1)dx}{x^3+x^2+x+1}$.

$$Ans.\ \log.\frac{(x+1)^{\frac{3}{2}}}{(x^2+1)^{\frac{1}{4}}}-\tfrac{1}{2}\tan g.^{-1}x.$$

Ex. 4. Integrate the expression $\frac{dx}{(x^2+1)(x^2+4)}$.

$$Ans.\ \tfrac{1}{3}\tan g.^{-1}x-\tfrac{1}{6}\tan g.^{-1}\frac{x}{2}.$$

177. Case IV. *When some of the binomial factors of the denominator are imaginary and equal.* In this case the product of each pair of imaginary factors is of the form $(x-a)^2+b^2$, and if there are n such factors, the denominator will contain a factor of the form $\{(x-a)^2+b^2\}^n$. In order to determine the partial fractions, we combine the methods employed in the

second and third cases, and assume n partial fractions of the form $\frac{Ax+B}{\{(x-a)^2+b^2\}^n}$, $\frac{Cx+D}{\{(x-a)^2+b^2\}^{n-1}}$, $\cdots$ $\frac{Mx+N}{(x-a)^2+b^2}$.

By clearing the equation of denominators, and proceeding as in Art. 176, the values of A, B, C, etc., may be determined.

Ex. 1. Integrate the expression $\frac{(x^3+x-1)dx}{(x^2+2)^2}$.

Assume $$\frac{x^3+x-1}{(x^2+2)^2}=\frac{Ax+B}{(x^2+2)^2}+\frac{Cx+D}{x^2+2}. \qquad (1)$$

Clearing of fractions, we find

$$x^3+x-1=Ax+B+Cx^3+Dx^2+2Cx+2D;$$

whence $1=C$, $0=D$, $1=A+2C$, $-1=B+2D$.

Therefore $A=-1$, $B=-1$, $C=1$, $D=0$.

Substituting these values in equation (1), and multiplying by dx, we have

$$\int\frac{(x^3+x-1)dx}{(x^2+2)^2}=-\int\frac{(x+1)dx}{(x^2+2)^2}+\int\frac{xdx}{x^2+2}$$

$$=-\int\frac{xdx}{(x^2+2)^2}-\int\frac{dx}{(x^2+2)^2}+\int\frac{xdx}{x^2+2}$$

$$=\frac{1}{2(x^2+2)}-\int\frac{dx}{(x^2+2)^2}+\tfrac{1}{2}\log.(x^2+2).$$

The term $\frac{-dx}{(x^2+2)^2}$ can be integrated by means of formula D, given in Art. 195.

Ex. 2. Integrate the expression $\frac{x^3dx}{\{(x-1)^2+1\}^2}$, or $\frac{x^3dx}{(x^2-2x+2)^2}$.

Assume $$\frac{x^3}{(x^2-2x+2)^2}=\frac{Ax+B}{(x^2-2x+2)^2}+\frac{Cx+D}{x^2-2x+2}. \qquad (1)$$

Clearing of fractions, and proceeding as in the last example, we find $A=2$, $B=-4$, $C=1$, and $D=2$.

Substituting these values in equation (1), and multiplying by dx, we have

$$\frac{x^3dx}{(x^2-2x+2)^2}=\frac{2xdx}{(x^2-2x+2)^2}-\frac{4dx}{(x^2-2x+2)^2}+\frac{xdx}{x^2-2x+2}+\frac{2dx}{x^2-2x+2}.$$

For convenience, substitute z for $x-1$, and we have

$$\frac{x^3dx}{(x^2-2x+2)^2}=\frac{2(z+1)dz}{(z^2+1)^2}-\frac{4dz}{(z^2+1)^2}+\frac{(z+1)dz}{z^2+1}+\frac{2dz}{z^2+1}$$

$$=\frac{2zdz}{(z^2+1)^2}-\frac{2dz}{(z^2+1)^2}+\frac{zdz}{z^2+1}+\frac{3dz}{z^2+1}.$$

Hence

$$\int\frac{x^3dx}{(x^2-3x+2)^2}=-\frac{1}{z^2+1}-2\int\frac{dz}{(z^2+1)^2}+\tfrac{1}{2}\log.(z^2+1)+3\,\text{tang.}^{-1}z.$$

The integral of the second term is found by formula D, Art. 195.

From the preceding discussion we infer that all differentials which are rational fractions can be integrated, provided we can discover the factors of the denominator. These integrals can all be found by methods already explained, with only one exception, to be considered in Art. 195.

INTEGRATION OF CERTAIN IRRATIONAL DIFFERENTIALS.

178. *When the irrational parts are monomials.* In the preceding articles, methods have been explained by which every rational differential can be integrated, with the exception of one case, referred to in Art. 177. We hence infer that *we can also integrate every irrational differential which can be made rational in terms of a new variable.*

The simplest irrational differentials are those in which all the irrational parts are *monomials.* Such expressions may be rendered rational by reducing the fractional exponents of the variable x to a common denominator n, and substituting z^n for x.

Ex. 1. Integrate the expression $\dfrac{1-x^{\frac{1}{2}}}{1-x^{\frac{1}{3}}}dx.$

The least common denominator of the exponents of x is 6. Hence we must substitute z^6 for x. Therefore $dx=6z^5dz$.

By substitution in the given expression, we have

$$\frac{1-x^{\frac{1}{2}}}{1-x^{\frac{1}{3}}}dx=\frac{1-z^3}{1-z^2}\cdot 6z^5dz=\frac{6z^8-6z^5}{z^2-1}dz,$$

which, by division, becomes

$$6\left(z^6+z^4-z^3+z^2-z+1-\frac{1}{1+z}\right)dz.$$

Hence

$$\int\left(\frac{1-x^{\frac{1}{2}}}{1-x^{\frac{1}{3}}}\right)dx=6\left\{\frac{z^7}{7}+\frac{z^5}{5}-\frac{z^4}{4}+\frac{z^3}{3}-\frac{z^2}{2}+z-\log.(1+z)\right\}$$

$$=6\left\{\frac{x^{\frac{7}{6}}}{7}+\frac{x^{\frac{5}{6}}}{5}-\frac{x^{\frac{2}{3}}}{4}+\frac{x^{\frac{1}{2}}}{3}-\frac{x^{\frac{1}{3}}}{2}+x^{\frac{1}{6}}-\log.(1+x^{\frac{1}{6}})\right\}+C.$$

Ex. 2. Integrate the expression $\frac{x^{\frac{1}{2}}-2x^{\frac{1}{3}}}{1+x^{\frac{1}{3}}}.dx.$

Assume $x=z^6$; then, by substitution, we have

$$\frac{x^{\frac{1}{2}}-2x^{\frac{1}{3}}}{1+x^{\frac{1}{3}}}.dx=\frac{6(z^8-2z^7)dz}{1+z^2},$$

which, by division, becomes

$$6\left(z^6-2z^5-z^4+2z^3+z^2-2z-1+\frac{2z+1}{z^2+1}\right)dz.$$

Hence

$$\int\frac{x^{\frac{1}{2}}-2x^{\frac{1}{3}}}{1+x^{\frac{1}{3}}}.dx=6\left\{\frac{z^7}{7}-\frac{2z^6}{6}-\frac{z^5}{5}+\frac{2z^4}{4}+\frac{z^3}{3}-\frac{2z^2}{2}-z+\log.(z^2+1)+\text{tang.}^{-1}z\right\}$$

$$=6\left\{\frac{x^{\frac{7}{6}}}{7}-\frac{x}{3}-\frac{x^{\frac{5}{6}}}{5}+\frac{x^{\frac{2}{3}}}{2}+\frac{x^{\frac{1}{2}}}{3}-x^{\frac{1}{3}}-x^{\frac{1}{6}}+\log.(x^{\frac{1}{3}}+1)+\text{tang.}^{-1}x^{\frac{1}{6}}\right\}.$$

Ex. 3. Integrate the expression $\frac{3x^{\frac{1}{2}}dx}{2x^{\frac{1}{2}}-x^{\frac{2}{3}}}.$

$$Ans.-18\left\{\frac{x^{\frac{5}{6}}}{5}+\frac{x^{\frac{2}{3}}}{2}+\frac{4x^{\frac{1}{2}}}{3}+4x^{\frac{1}{3}}+16x^{\frac{1}{6}}+32\log.(2-x^{\frac{1}{6}})\right\}+C.$$

Ex. 4. Integrate the expression $\frac{2x^{\frac{1}{2}}-3x^{\frac{2}{3}}}{5x^{\frac{1}{6}}}.dx.$

$$Ans.\ \frac{3}{10}x^{\frac{4}{3}}-\frac{2}{5}x^{\frac{3}{2}}+C.$$

179. *When the irrational parts are binomials.* When all the irrational parts of the differential expressions are of the form $(a+bx)^{\frac{r}{q}}$; that is, consist of fractional powers of the same binomial of the first degree, the expression may be rendered rational in terms of z by regarding $a+bx$ as the variable, and assuming $a+bx=z^n$, where n represents the least common denominator of the exponents of the radicals.

Ex. 1. Integrate the expression $\dfrac{xdx}{(a+bx)^{\frac{1}{2}}}$.

Assume $a+bx=z^2$; then, by substitution, we have

$$\frac{xdx}{(a+bx)^{\frac{1}{2}}}=\frac{2(z^2-a)dz}{b^2}=\frac{2z^2dz}{b^2}-\frac{2adz}{b^2};$$

$$\int\frac{xdx}{(a+bx)^{\frac{1}{2}}}=\frac{2z^3}{3b^2}-\frac{2az}{b^2}=\frac{2z}{3b^2}(z^2-3a)$$

$$=\frac{2(a+bx)^{\frac{1}{2}}}{3b^2}(bx-2a)+\mathrm{C}.$$

Ex. 2. Integrate the expression $\dfrac{dx}{(1+x)^{\frac{3}{2}}+(1+x)^{\frac{1}{2}}}$.

Assume $1+x=z^2$; then

$$\frac{dx}{(1+x)^{\frac{3}{2}}+(1+x)^{\frac{1}{2}}}=\frac{2zdz}{z^3+z}=\frac{2dz}{1+z^2};$$

$$\int\frac{dx}{(1+x)^{\frac{3}{2}}+(1+x)^{\frac{1}{2}}}=2\int\frac{dz}{1+z^2}=2\ \text{tang.}^{-1}z=2\ \text{tang.}^{-1}(1+x)^{\frac{1}{2}}+\mathrm{C}.$$

Ex. 3. Integrate the expression $\dfrac{xdx}{(1+x)^{\frac{1}{3}}-(1+x)^{\frac{1}{2}}}$.

Assume $1+x=z^6$.

$$\frac{xdx}{(1+x)^{\frac{1}{3}}-(1+x)^{\frac{1}{2}}}=\frac{(z^6-1)6z^5dz}{z^2-z^3}=-6(z^8+z^7+z^6+z^5+z^4+z^3)dz;$$

$$\int\frac{xdx}{(1+x)^{\frac{1}{3}}-(1+x)^{\frac{1}{2}}}=-6\left(\frac{z^9}{9}+\frac{z^8}{8}+\frac{z^7}{7}+\frac{z^6}{6}+\frac{z^5}{5}+\frac{z^4}{4}\right)+\mathrm{C}.$$

Ex. 4. Integrate the expression $\frac{dx}{(a+bx)^{\frac{1}{2}}}$.

$$Ans.\ \frac{2(a+bx)^{\frac{1}{2}}}{b}+C.$$

180. *When the irrational parts are trinomials.* When the expression to be integrated contains no radical quantity but one of the form $\sqrt{a+bx\pm cx^2}$, it may be rendered rational by substitution. If we remove the factor c from under the radical sign, the above expression may be written $\sqrt{c}\sqrt{\frac{a}{c}+\frac{bx}{c}\pm x^2}$, or $\sqrt{c}\sqrt{a'+b'x\pm x^2}$. It is therefore only necessary to show how an expression containing the radical $\sqrt{a+bx\pm x^2}$ may be rendered rational.

Case I. *When the sign of x^2 is plus.*

Assume $$a+bx+x^2=(z-x)^2.$$

Then $$a+bx=z^2-2zx,$$

or $$x=\frac{z^2-a}{b+2z}.$$

Differentiating this value of x, we obtain

$$dx=\frac{2(z^2+bz+a)dz}{(b+2z)^2};$$

also, $$\sqrt{a+bx+x^2}=z-x=\frac{z^2+bz+a}{b+2z}.$$

These values of x, dx, and $\sqrt{a+bx+x^2}$, substituted in the given differential expression, will furnish a rational expression in terms of z and dz, which can therefore be integrated.

Ex. 1. Integrate the expression $\frac{dx}{\sqrt{1+x+x^2}}$.

Assume $$1+x+x^2=(z-x)^2.$$

Then $$x=\frac{z^2-1}{2z+1}, \text{ and } dx=\frac{2(z^2+z+1)dz}{(2z+1)^2};$$

$$\sqrt{1+x+x^2}=z-x=\frac{z^2+z+1}{2z+1}.$$

Hence $$\frac{dx}{\sqrt{1+x+x^2}}=\frac{2(z^2+z+1)}{(2z+1)^2}\times\frac{2z+1}{z^2+z+1}\cdot dz=\frac{2dz}{2z+1};$$

$$\int\frac{dx}{\sqrt{1+x+x^2}}=\log.(2z+1)=\log.(2x+1+2\sqrt{1+x+x^2})+C.$$

Ex. 2. Integrate the expression $\frac{dx}{x\sqrt{1+x+x^2}}$.

Assume $$1+x+x^2=(z-x)^2.$$

$$\frac{dx}{x\sqrt{1+x+x^2}}=\frac{2(z^2+z+1)}{(2z+1)^2}\times\frac{2z+1}{z^2-1}\times\frac{2z+1}{z^2+z+1}\cdot dz$$

$$=\frac{2dz}{z^2-1}=\frac{dz}{z-1}-\frac{dz}{z+1};$$

$$\int\frac{dx}{x\sqrt{1+x+x^2}}=\log.\frac{z-1}{z+1}=\log.\frac{x-1+\sqrt{1+x+x^2}}{x+1+\sqrt{1+x+x^2}};$$

or, multiplying both numerator and denominator by

$$x-1-\sqrt{1+x+x^2},$$

we obtain $$\log.\frac{3x}{x+2+2\sqrt{1+x+x^2}}+C.$$

181. Case II. *When the sign of x^2 is minus.* In this case the preceding method can not be employed; for if we assume $a+bx-x^2=(z-x)^2$, the second powers of x in the two members of the equation will not cancel each other, and the value of x thence deduced will not be rational in terms of z.

Let p and q be the two roots of the equation

$$x^2-bx-a=0;$$

then, by Algebra, Art. 434,

$$x^2-bx-a=(x-p)(x-q);$$

or, changing the signs,

$$a+bx-x^2=(x-p)(q-x).$$

Assume $$\sqrt{(x-p)(q-x)}=(x-p)z.$$

Squaring both members, and canceling the common factor $x-p$, we have $$q-x=(x-p)z^2.$$

Whence $$x=\frac{q+pz^2}{z^2+1};\ dx=\frac{2(p-q)zdz}{(z^2+1)^2};$$

$$x-p=\frac{q+pz^2}{z^2+1}-p=\frac{q-p}{z^2+1};\ \sqrt{(x-p)(q-x)}=\frac{(q-p)z}{z^2+1}.$$

These values of x, dx, and $\sqrt{a+bx-x^2}$, substituted in the original expression, will furnish a rational expression in terms of z and dz, which can therefore be integrated.

Ex. 1. Integrate the expression $\frac{dx}{\sqrt{1+x-x^2}}$.

The roots of the equation $x^2-x-1=0$ are

$$\tfrac{1}{2}+\tfrac{1}{2}\sqrt{5},\text{ and }\tfrac{1}{2}-\tfrac{1}{2}\sqrt{5}.$$

Substituting the values of dx and $\sqrt{1+x-x^2}$ before found, we have

$$\frac{dx}{\sqrt{1+x-x^2}}=\frac{2(p-q)zdz}{(z^2+1)^2}\times\frac{z^2+1}{(q-p)z}=-\frac{2dz}{z^2+1}.$$

Hence

$$\int\frac{dx}{\sqrt{1+x-x^2}}=-\int\frac{2dz}{z^2+1}=-2\text{ tang.}^{-1}z=-2\text{ tang.}^{-1}\sqrt{\frac{\frac{1}{2}-\frac{1}{2}\sqrt{5}-x}{x-\frac{1}{2}-\frac{1}{2}\sqrt{5}}}+\text{C}.$$

Ex. 2. Integrate the expression $\frac{dx}{\sqrt{2-x-x^2}}$.

Ans. $-2\text{ tang.}^{-1}\sqrt{\frac{1-x}{x+2}}+\text{C}.$

CHAPTER III.

INTEGRATION OF BINOMIAL DIFFERENTIALS. INTEGRATION BY PARTS. FORMULAS OF REDUCTION.

182. *Binomial differential defined.* A binomial differential is a differential expression which contains a binomial factor involving the unknown quantity. If the expression be rational, its integral can be found by the method of Art. 173. We have now to consider the case in which the expression is irrational, as $(3x^{\frac{2}{3}}+5x^{\frac{3}{4}})^{\frac{5}{6}}dx$, or $(ax^r+bx^n)^{\frac{p}{q}}dx$. Such expressions can generally be reduced to a simpler form.

183. *How a binomial differential can be reduced to a simpler form.*

1. If the variable x were found in both terms of the binomial, and the expression were of the form

$$(ax^r+bx^n)^{\frac{p}{q}}dx,$$

we may divide the binomial within the parenthesis by x^r (r being supposed less than n), and placing this factor without the parenthesis, we shall have

$$x^{\frac{pr}{q}}(a+bx^{n-r})^{\frac{p}{q}}dx,$$

in which but one of the terms within the parenthesis contains the variable x, and the exponent of this variable is positive.

2. If after this reduction the exponent of x, either within or without the parenthesis, or both, should be fractional, we can substitute for x another variable with an exponent equal to the least common multiple of the denominators of the given exponents, by which means the proposed binomial will be transformed into one in which the exponents of the variable are whole numbers.

Thus in the example

$$x^{\frac{1}{3}}(a+bx^{\frac{1}{2}})^{\frac{p}{q}}dx,$$

if we make $x=z^6$, we obtain

$$x^{\frac{1}{3}}(a+bx^{\frac{1}{2}})^{\frac{p}{q}}dx=6z^7(a+bz^3)^{\frac{p}{q}}dz,$$

in which the exponents of z are whole numbers.

Hence *every binomial differential can be reduced to the form*

$$x^{m-1}(a+bx^n)^{\frac{p}{q}}dx, \qquad (1)$$

in which m and n are whole numbers, and n is positive.

184. *When this binomial can be integrated.* If the exponent of the parenthesis in binomial (1), Art. 183, is a whole number and positive, the expression can be integrated by raising the quantity within the parenthesis to the proposed power, multiplying each term by the factor without the parenthesis, and then integrating each term separately (Art. 160).

If $\frac{p}{q}$ is a whole number and negative, we have

$$x^{m-1}(a+bx^n)^{\frac{-p}{q}}dx=\frac{x^{m-1}dx}{(a+bx^n)^{\frac{p}{q}}},$$

which is a rational fraction, and can be integrated by Arts. 173–177.

185. *Every binomial differential of the form* (1), Art. 183, *can be integrated when* $\frac{m}{n}$ *is a whole number.*

For this purpose we substitute for the binomial within the parenthesis a new variable having an exponent equal to the denominator of the exponent of the parenthesis.

Assume $a+bx^n=z^q$;

then $(a+bx^n)^{\frac{p}{q}}=z^p$. (1)

Also, $x^n=\frac{z^q-a}{b}$,

and $x^m=\left(\frac{z^q-a}{b}\right)^{\frac{m}{n}}$.

Differentiating, we have

$$mx^{m-1}dx=\frac{mq}{nb}z^{q-1}\left(\frac{z^q-a}{b}\right)^{\frac{m}{n}-1}dz. \qquad (2)$$

Multiplying together equations (1) and (2), and dividing by m, we obtain

$$x^{m-1}(a+bx^n)^{\frac{p}{q}}dx=\frac{q}{nb}z^{p+q-1}\left(\frac{z^q-a}{b}\right)^{\frac{m}{n}-1}dz.$$

which, according to Art. 184, can be integrated when $\frac{m}{n}$ is a whole number.

Ex. 1. Integrate the expression

$$du=x^3(a+bx^2)^{\frac{3}{2}}dx.$$

Assume $$a+bx^2=z^2.$$

Then $$(a+bx^2)^{\frac{3}{2}}=z^3; \qquad (1)$$

also, $$x^2=\frac{z^2-a}{b}, \qquad (2)$$

and $$xdx=\frac{zdz}{b}. \qquad (3)$$

Multiplying together equations (1), (2), and (3), we obtain

$$du=x^5(a+bx^2)^{\frac{3}{2}}dx=\frac{z^6-az^4}{b^2}.dz.$$

Hence $$u=\frac{z^7}{7b^2}-\frac{az^5}{5b^2}=\frac{5z^2-7a}{35b^2}\times z^5.$$

Replacing the value of z, we find

$$u=\frac{5bx^2-2a}{35b^2}(a+bx^2)^{\frac{5}{2}}+\mathrm{C}.$$

Ex. 2. Integrate the expression $du=x^5(a+bx^2)^{\frac{1}{2}}dx$.

Assume $$a+bx^2=z^2.$$

Then $$(a+bx^2)^{\frac{1}{2}}=z, \qquad (1)$$

$$x^2=\frac{z^2-a}{b}, \qquad (2)$$

and $$xdx=\frac{zdz}{b}. \qquad (3)$$

Hence $$du=\left(\frac{z^2-a}{b}\right)^2\frac{z^2dz}{b}$$

$$=\frac{z^6-2az^4+a^2z^2}{b^3}.dz.$$

Therefore

$$u=\frac{z^7}{7b^3}-\frac{2az^5}{5b^3}+\frac{a^2z^3}{3b^3}$$

$$=\frac{z^3}{105b^3}(15z^4-42az^2+35a^2)$$

$$=\frac{(a+bx^2)^{\frac{3}{2}}}{105b^3}\{15(a+bx^2)^2-42a(a+bx^2)+35a^2\}+C.$$

Ex. 3. Integrate the expression $du=x^3(1+x^2)^{\frac{1}{2}}dx.$

$$Ans.\ u=(1+x^2)^{\frac{3}{2}}\frac{(3x^2-2)}{15}+C.$$

Ex. 4. Integrate the expression $du=x^2(a+x)^{\frac{1}{2}}dx.$

$$Ans.\ u=\frac{2(a+x)^{\frac{3}{2}}}{105}(15x^2-12ax+8a^2)+C.$$

Ex. 5. Integrate the expression $du=x^5(a+bx^2)^{\frac{2}{3}}dx.$

$$Ans.\ u=\frac{3(a+bx^2)^{\frac{11}{3}}}{22b^3}-\frac{3a(a+bx^2)^{\frac{8}{3}}}{8b^3}+\frac{3a^2(a+bx^2)^{\frac{5}{3}}}{10b^3}+C.$$

Ex. 6. Integrate the expression $du=x^3(a-x^2)^{-\frac{1}{2}}dx.$

$$Ans.\ u=-\tfrac{1}{3}(a-x^2)^{\frac{1}{2}}(2a+x^2)+C.$$

Ex. 7. Integrate the expression $du=x^5(a^2+x^2)^{-1}dx.$

Assume $$a^2+x^2=z.$$

$$du=\left(\frac{z}{2}-a^2+\frac{a^4}{2z}\right)dz.$$

$$u=\frac{z^2}{4}-a^2z+\frac{a^4}{2}\log.z$$

$$=\left(\frac{a^2+x^2}{2}\right)^2-a^2(a^2+x^2)+\frac{a^4}{2}\log.(a^2+x^2)+C.$$

Ex. 8. Integrate the expression $x^3(a+bx^2)^{\frac{1}{2}}dx$.

$$Ans. \frac{3bx^2-2a}{15b^2}(a+bx^2)^{\frac{3}{2}}.$$

Ex. 9. Integrate the expression $x^3(a^2+x^2)^{\frac{1}{3}}dx$.

$$Ans. \frac{3}{56}(a^2+x^2)^{\frac{4}{3}}(4x^2-3a^2).$$

Ex. 10. Integrate the expression $\frac{x^3dx}{\sqrt{a+bx^2}}$.

$$Ans. \frac{bx^2-2a}{3b^2}(a+bx^2)^{\frac{1}{2}}.$$

186. *Every binomial differential of the form* (1), *Art.* 183, *can be integrated when* $\frac{m}{n}+\frac{p}{q}$ *is a whole number.*

The binomial $x^{m-1}(a+bx^n)^{\frac{p}{q}}dx$ may be written

$$x^{m-1}\left\{\left(\frac{a}{x^n}+b\right)x^n\right\}^{\frac{p}{q}}dx,$$

or

$$x^{m-1}(ax^{-n}+b)^{\frac{p}{q}}x^{\frac{np}{q}}dx,$$

which is equal to $x^{m+\frac{np}{q}-1}(ax^{-n}+b)^{\frac{p}{q}}dx$,

which, according to Art. 185, can be integrated when

$$\frac{m+\frac{np}{q}}{n} \text{ is a whole number,}$$

or

$$\frac{m}{n}+\frac{p}{q} \text{ is a whole number.}$$

Ex. 1. Integrate the expression $du=a(1+x^2)^{-\frac{3}{2}}dx$.

Assume $x^{-2}+1=v^2$;

that is, $1+x^2=v^2x^2$.

Then $(1+x^2)^{-\frac{3}{2}}=v^{-3}x^{-3}$; (1)

also, $x^2=\frac{1}{v^2-1}$.

Whence $$dx=\frac{-vdv}{x(v^2-1)^2}, \tag{2}$$

and $$1=x^4(v^2-1)^2. \tag{3}$$

Multiplying together equations (1), (2) and (3), we have

$$du=a(1+x^2)^{-\frac{3}{2}}dx=-\frac{adv}{v^2}.$$

Whence $$u=\frac{a}{v}=\frac{ax}{\sqrt{1+x^2}}+\text{C}.$$

Ex. 2. Integrate the expression $du=x^{-4}(1+x^2)^{-\frac{1}{2}}dx$.

Assume $$1+x^2=v^2x^2.$$

Then $$x^{-2}=v^2-1,$$

$$x^{-4}=(v^2-1)^2; \tag{1}$$

also, $$x=(v^2-1)^{-\frac{1}{2}}.$$

Whence $$dx=-(v^2-1)^{-\frac{3}{2}}vdv; \tag{2}$$

also, $$(1+x^2)^{-\frac{1}{2}}=\frac{1}{vx}=\frac{(v^2-1)^{\frac{1}{2}}}{v}. \tag{3}$$

Multiplying together equations (1), (2), and (3), we have

$$du=x^{-4}(1+x^2)^{-\frac{1}{2}}dx=-(v^2-1)dv.$$

Whence $$u=-\frac{v^3}{3}+v=\frac{(2x^2-1)(1+x^2)^{\frac{1}{2}}}{3x^3}+\text{C}.$$

Ex. 3. Integrate the expression $x^{-4}(1-2x^2)^{-\frac{1}{2}}dx$.

$$Ans. -\frac{1+4x^2}{3x^3}(1-2x^2)^{\frac{1}{2}}+\text{C}$$

Ex. 4. Integrate the expression $du=x^{-2}(1+x^2)^{-\frac{3}{2}}dx$.

$$Ans. -\left(2x+\frac{1}{x}\right)(1+x^2)^{-\frac{1}{2}}+\text{C}.$$

Ex. 5. Integrate the expression $du=(a^2+x^2)^{-\frac{1}{2}}dx$.

Assume $$v=x+(a^2+x^2)^{\frac{1}{2}}.$$

Then $$dv=dx+x(a^2+x^2)^{-\frac{1}{2}}dx=\frac{x+(a^2+x^2)^{\frac{1}{2}}}{(a^2+x^2)^{\frac{1}{2}}}dx.$$

Therefore $$\frac{dv}{v}=(a^2+x^2)^{-\frac{1}{2}}dx.$$

Hence $$u=\log. v=\log.\{x+(a^2+x^2)^{\frac{1}{2}}\}+\text{C}.$$

Ex. 6. Integrate the expression $\frac{2adx}{x\sqrt{2ax-x^2}}$.

$$Ans. -\frac{2\sqrt{2ax-x^2}}{x}.$$

Ex. 7. Integrate the expression $\frac{xdx}{(2ax-x^2)^{\frac{3}{2}}}$. $Ans. \frac{\sqrt{x}}{a\sqrt{2a-x}}$.

Ex. 8. Integrate the expression $\frac{dx}{x^2\sqrt{a^2+x^2}}$.

$$Ans. -\frac{\sqrt{a^2+x^2}}{a^2x}.$$

Ex. 9. Integrate the expression $x^{-2}(a+x^3)^{-\frac{5}{3}}dx$.

$$Ans. -\frac{3x^3+2a}{2a^2x(a+x^3)^{\frac{2}{3}}}.$$

INTEGRATION BY PARTS.

187. The integral of a differential expression may sometimes be made to depend upon the integral of another differential of a simpler form. This is effected by resolving the expression into two parts, one of which has a known integral.

If u and v are functions of x, then, by Art. 24,

$$d(uv)=udv+vdu.$$

Whence $$uv=\int udv+\int vdu,$$

and consequently $$\int udv=uv-\int vdu; \qquad (1)$$

from which we see that the integral of udv can be obtained, provided we are able to integrate vdu. This method of integrating udv is called *integrating by parts.*

To apply this method to a particular example, we resolve the given differential into two factors, one of which shall contain the differential of the variable, and be capable of integration. We substitute this factor for dv in formula (1), we substitute its integral for v, and we substitute the other factor for u.

Ex. 1. Integrate the expression $x^3(1+x^2)^4dx$, or $x^2(1+x^2)^4xdx$.

This expression can be resolved into the two factors x^2 and

$(1+x^2)^4 x dx$, the latter of which is the differential of a known function.

Hence we put

$$dv=(1+x^2)^4 x dx, \text{ whence } v=\frac{1}{10}(1+x^2)^5;$$

$$u=x^2, \text{ whence } du=2x dx.$$

Hence $\int u dv=\int x^2(1+x^2)^4 x dx=uv-\int v du$

$$=\frac{x^2}{10}(1+x^2)^5-\int\frac{1}{10}(1+x^2)^5 2x dx$$

$$=\frac{x^2}{10}(1+x^2)^5-\frac{1}{60}(1+x^2)^6$$

$$=\frac{1}{60}(1+x^2)^5(5x^2-1).$$

Ex. 2. Integrate the expression $\frac{x^3 dx}{(1+x^2)^{\frac{3}{2}}}$, or $x^2(1+x^2)^{-\frac{3}{2}} x dx$.

This expression can be resolved into the two factors x^2 and $(1+x^2)^{-\frac{3}{2}} x dx$, the latter of which can be integrated.

Hence we put

$$dv=(1+x^2)^{-\frac{3}{2}} x dx;\ v=-(1+x^2)^{-\frac{1}{2}};$$

$$u=x^2;\ du=2x dx.$$

Hence $\int u dv=\int\frac{x^3 dx}{(1+x^2)^{\frac{3}{2}}}=uv-\int v du$

$$=\frac{-x^2}{(1+x^2)^{\frac{1}{2}}}+\int\frac{2x dx}{(1+x^2)^{\frac{1}{2}}}$$

$$=\frac{-x^2}{(1+x^2)^{\frac{1}{2}}}+2(1+x^2)^{\frac{1}{2}}$$

$$=\frac{x^2+2}{(1+x^2)^{\frac{1}{2}}}.$$

Ex. 3. Integrate the expression $x \log. x dx$.

Put

$$dv=x dx,\ v=\frac{x^2}{2},$$

$$u=\log. x,\ du=\frac{dx}{x}.$$

Hence $$\int x \log. x dx = uv - \int v du$$
$$= \log. x . \frac{x^2}{2} - \int \frac{x^2}{2} \cdot \frac{dx}{x}$$
$$= \frac{x^2}{2} \log. x - \frac{x^2}{4}$$
$$= \frac{x^2}{2}\left(\log. x - \frac{1}{2}\right).$$

Ex. 4. Integrate the expression $\frac{x^3 dx}{(1+x^2)^{\frac{5}{2}}}$. *Ans.* $-\frac{3x^2+2}{3(1+x^2)^{\frac{5}{2}}}$.

Ex. 5. Integrate the expression $x^n \log. x dx$.

Put $u = \log. x$, and $dv = x^n dx$.

$$\int x^n \log. x dx = \frac{x^{n+1} \log. x}{n+1} - \int \frac{x^n dx}{n+1}$$
$$= \frac{x^{n+1} \log. x}{n+1} - \frac{x^{n+1}}{(n+1)^2}.$$

Ex. 6. Integrate the expression $x^3(a-x^2)^{\frac{1}{2}} dx$.

Put $u = x^2$, and $dv = x(a-x^2)^{\frac{1}{2}} dx$.

$$\int x^3 (a-x^2)^{\frac{1}{2}} dx = -\frac{x^2}{3}(a-x^2)^{\frac{3}{2}} + \int \frac{(a-x^2)^{\frac{3}{2}}}{3} . 2x dx$$
$$= -\frac{x^2}{3}(a-x^2)^{\frac{3}{2}} - \frac{2}{15}(a-x^2)^{\frac{5}{2}}.$$

Ex. 7. Integrate the expression $\frac{(1-x^2)^{\frac{1}{2}} dx}{x^2}$.

Put $u = (1-x^2)^{\frac{1}{2}}$, and $dv = \frac{dx}{x^2}$.

$$\int \frac{(1-x^2)^{\frac{1}{2}} dx}{x^2} = -\frac{(1-x^2)^{\frac{1}{2}}}{x} - \int \frac{dx}{(1-x^2)^{\frac{1}{2}}}$$
$$= -\frac{(1-x^2)^{\frac{1}{2}}}{x} - \sin.^{-1} x \text{ (see Art. 168).}$$

Ex. 8. Integrate the expression $(a^2-x^2)^{\frac{1}{2}} dx$.

Put $u = (a^2-x^2)^{\frac{1}{2}}$, and $dv = dx$.

$$\int (a^2-x^2)^{\frac{1}{2}}dx = x(a^2-x^2)^{\frac{1}{2}} + \int \frac{x^2dx}{(a^2-x^2)^{\frac{1}{2}}}. \qquad (1)$$

But $$\int (a^2-x^2)^{\frac{1}{2}}dx = \int \frac{a^2-x^2}{(a^2-x^2)^{\frac{1}{2}}}dx$$

$$= \int \frac{a^2dx}{(a^2-x^2)^{\frac{1}{2}}} - \int \frac{x^2dx}{(a^2-x^2)^{\frac{1}{2}}}. \qquad (2)$$

Adding equations (1) and (2), we have

$$2\int (a^2-x^2)^{\frac{1}{2}}dx = x(a^2-x^2)^{\frac{1}{2}} + a^2\int \frac{dx}{(a^2-x^2)^{\frac{1}{2}}}.$$

Hence $\int (a^2-x^2)^{\frac{1}{2}}dx = \frac{x}{2}(a^2-x^2)^{\frac{1}{2}} + \frac{a^2}{2}\sin.^{-1}\frac{x}{a}$ (see Art. 168).

188. By the application of the preceding method many binomial differentials can be integrated. For greater convenience, we shall henceforth represent any binomial differential by the form $x^m(a+bx^n)^p dx$,
where p is supposed to be a fraction, but m and n are whole numbers, and n is positive.

METHOD OF REDUCTION.

189. This method consists in making the integral of the proposed expression depend upon the integral of another differential of the same form, but in which the exponents are diminished, so that by repeating the process we at length obtain a differential expression which can be integrated by methods already established; or, in case integration is not possible, we have the part not integrated in the simplest form possible.

190. *The integral of any differential of the form*

$$x^m(a+bx^n)^p dx$$

may be made to depend upon the integral of

$$x^{m-n}(a+bx^n)^{p+1}dx.$$

Assume $dv = (a+bx^n)^p x^{n-1}dx$; whence $u = x^{m-n+1}$.

Integrating the former expression, and differentiating the latter, we have

$$v=\frac{(a+bx^n)^{p+1}}{nb(p+1)},$$

and

$$du=(m-n+1)x^{m-n}dx.$$

Substituting these values in the formula $\int udv=uv-\int vdu$, we have

$$\int x^m(a+bx^n)^p dx$$

$$=\frac{x^{m-n+1}(a+bx^n)^{p+1}}{nb(p+1)}-\frac{m-n+1}{nb(p+1)}\int x^{m-n}(a+bx^n)^{p+1}dx. \quad (1)$$

Equation (1) is called a *formula of reduction*, and by means of it we make the integral of $x^m(a+bx^n)^p$ depend on that of $x^{m-n}(a+bx^n)^{p+1}$. In the same way the latter integral can be made to depend on that of $x^{m-2n}(a+bx^n)^{p+2}$, and so on. For example, if $m=2$, $n=2$, and $p=-\frac{3}{2}$, we have

$$\int\frac{x^2dx}{(a+bx^2)^{\frac{3}{2}}}=-\frac{x}{b\sqrt{a+bx^2}}+\frac{1}{b}\int\frac{dx}{\sqrt{a+bx^2}}.$$

The integral of the latter expression can be found, and hence the integral of the former expression becomes known.

191. *The integral of any differential of the form*

$$x^m(a+bx^n)^p dx$$

may be made to depend upon the integral of

$$x^{m+n}(a+bx^n)^{p-1}dx.$$

From Equation (1), Art. 190, by transposition, we obtain

$$\int x^{m-n}(a+bx^n)^{p+1}dx=\frac{x^{m-n+1}(a+bx^n)^{p+1}}{m-n+1}-\frac{nb(p+1)}{m-n+1}\int x^m(a+bx^n)^p dx.$$

For greater convenience, substitute m for $m-n$, and p for $p+1$, and we have

$$\int x^m(a+bx^n)^p dx=\frac{x^{m+1}(a+bx^n)^p}{m+1}-\frac{bnp}{m+1}\int x^{m+n}(a+bx^n)^{p-1}dx. \quad (2)$$

By means of Eq. (2) we make the integral of $x^m(a+bx^n)^p dx$ depend on that of $x^{m+n}(a+bx^n)^{p-1}dx$. In the same way the latter integral can be made to depend on that of $x^{m+2n}(a+bx^n)^{p-2}dx$;

and by repeated applications of the formula the exponent of the binomial, if positive, may be reduced to a fraction less than unity, either positive or negative.

192. *Modification of formulas* (1) *and* (2). Equations (1) and (2), Arts. 190 and 191, may be reduced to forms which are more convenient for use.

Since

$$(a+bx^n)^{p+1}=(a+bx^n)^p(a+bx^n)=a(a+bx^n)^p+bx^n(a+bx^n)^p,$$

if we multiply by $x^{m-n}dx$, we shall obtain

$$x^{m-n}(a+bx^n)^{p+1}dx=ax^{m-n}(a+bx^n)^pdx+bx^m(a+bx^n)^pdx.$$

Substitute this value in the last term of equation (1), Art. 190, multiply the result by $nb(p+1)$, transpose the last term of the product to the first member of the equation, and divide the equation thus obtained by $b(np+m+1)$, and we shall obtain

FORMULA A.

$$\int x^m(a+bx^n)^pdx=\frac{x^{m-n+1}(a+bx^n)^{p+1}-a(m-n+1)\int x^{m-n}(a+bx^n)^pdx}{b(np+m+1)}.$$

By means of Formula A the integral of the proposed differential is made to depend upon the integral of another differential in which *the exponent of the factor* x^m *without the parenthesis is diminished by that of* x^n, *the variable within the parenthesis.* By a second application of the same formula, we should find $\int x^{m-n}(a+bx^n)^pdx$ to depend upon $\int x^{m-2n}(a+bx^n)^pdx$; and, by continuing the process, the exponent of the factor without the parenthesis may be diminished until it is less than n. If m is positive, and if $m+1$ is an exact multiple of n, then, after a certain number of applications of the formula, the integration will be entirely accomplished.

193. *When the exponent* m *is negative.* By the application of Formula A the exponent m will be numerically diminished only when m is positive; but from this formula we may deduce another, which will numerically diminish the exponent

when m is negative. For this purpose, clear Formula A of fractions, transpose the term which does not contain the sign of integration, divide by $a(m-n+1)$, and we obtain

$$\int x^{m-n}(a+bx^n)^p dx = \frac{x^{m-n+1}(a+bx^n)^{p+1} - b(np+m+1)\int x^m(a+bx^n)^p dx}{a(m-n+1)}.$$

For greater convenience, substitute $m+n$ for m, and we obtain

FORMULA B.

$$\int x^m(a+bx^n)^p dx = \frac{x^{m+1}(a+bx^n)^{p+1} - b(np+m+n+1)\int x^{m+n}(a+bx^n)^p dx}{a(m+1)}.$$

By the application of this formula, when m is negative, m will be numerically diminished by the number of units in n. If $np+m+n+1=0$, or $n=-\frac{m+1}{p+1}$, the part to be integrated will disappear, and the integration will be complete.

194. *To diminish the exponent of the parenthesis.*

Since

$$(a+bx^n)^p = (a+bx^n)^{p-1}(a+bx^n) = a(a+bx^n)^{p-1} + bx^n(a+bx^n)^{p-1},$$

multiplying each member by $x^m dx$, we obtain

$$x^m(a+bx^n)^p dx = ax^m(a+bx^n)^{p-1}dx + bx^{m+n}(a+bx^n)^{p-1}dx;$$

and integrating, we have

$$\int x^m(a+bx^n)^p dx = a\int x^m(a+bx^n)^{p-1}dx + b\int x^{m+n}(a+bx^n)^{p-1}dx.$$

Transposing the terms of Equation (2), Art. 191, we obtain

$$\int x^{m+n}(a+bx^n)^{p-1}dx = \frac{x^{m+1}(a+bx^n)^p}{bnp} - \frac{m+1}{bnp}\int x^m(a+bx^n)^p dx.$$

Hence, by substitution, we have

$$\int x^m(a+bx^n)^p dx = a\int x^m(a+bx^n)^{p-1}dx + \frac{x^{m+1}(a+bx^n)^p}{np} - \frac{m+1}{np}\int x^m(a+bx^n)^p dx.$$

Transposing and uniting terms, we obtain

FORMULA C.

$$\int x^m(a+bx^n)^p dx = \frac{x^{m+1}(a+bx^n)^p + anp\int x^m(a+bx^n)^{p-1}dx}{np+m+1}.$$

By this formula the value of the required integral is made

to depend upon another in which the exponent of the binomial is diminished by unity. By a second application of the same formula the value of this new integral may be made to depend upon that of an integral in which the exponent of the binomial is still further diminished; and by repeated applications of the formula the exponent of the binomial, if positive, may be reduced to a fraction less than unity, either positive or negative.

195. *When the exponent p is negative.* By the application of Formula C the exponent p will be numerically diminished only when p is positive; but from this formula we may deduce another, which will numerically diminish the exponent when p is negative. For this purpose, clear Formula C of fractions, transpose the term which does not contain the sign of integration, divide by anp, and we have

$$\int x^m(a+bx^n)^{p-1}dx = \frac{-x^{m+1}(a+bx^n)^p + (np+m+1)\int x^m(a+bx^n)^p dx}{anp}.$$

For greater convenience, substitute $p+1$ for p, and we obtain

FORMULA D.

$$\int x^m(a+bx^n)^p dx = \frac{-x^{m+1}(a+bx^n)^{p+1} + (np+n+m+1)\int x^m(a+bx^n)^{p+1}dx}{an(p+1)}.$$

By the application of this formula, when p is negative, p will be numerically diminished by unity. If $p=-1$, the second member of the formula becomes infinite; but in this case the primitive expression is a rational fraction, and may be integrated by Arts. 173–177. If $np+n+m+1=0$, or $n=-\frac{m+1}{p+1}$, the part to be integrated will disappear, and the integration will be complete.

196. *Modification of Formula A.* We have frequent occasion to integrate binomial differentials of the form

$$\frac{x^m dx}{\sqrt{2ax-x^2}}.$$

If we remove the factor x from under the radical sign, this expression may be written $x^{m-\frac{1}{2}}(2a-x)^{-\frac{1}{2}}dx$, which may be integrated by Formula A, by making the proper substitutions. Substitute $m-\frac{1}{2}$ for m, $2a$ for a, -1 for b, 1 for n, and $-\frac{1}{2}$ for p, and we have

$$\int x^{m-\frac{1}{2}}(2a-x)^{-\frac{1}{2}}dx = \frac{x^{m-\frac{1}{2}}(2a-x)^{\frac{1}{2}} - 2a(m-\frac{1}{2})\int x^{m-\frac{3}{2}}(2a-x)^{-\frac{1}{2}}dx}{-m}.$$

or, changing the form, we derive

FORMULA E.

$$\int \frac{x^m dx}{\sqrt{2ax-x^2}} = -\frac{x^{m-1}\sqrt{2ax-x^2}}{m} + \frac{a(2m-1)}{m}\int \frac{x^{m-1}dx}{\sqrt{2ax-x^2}}.$$

By the application of this formula the exponent m is diminished by unity. If m is entire and positive, by repeated applications of the formula we ultimately arrive at the form

$$\int \frac{dx}{\sqrt{2ax-x^2}},$$

which is equal to the arc of a circle whose versed sine is $\frac{x}{a}$ (see Art. 170).

197. *Application of the preceding formulas.* By the application of Formulas A, B, C, D, and E, a great variety of differential expressions may be integrated.

Ex. 1. Integrate the expression $x^3(a+bx^2)^{-\frac{1}{2}}dx$.

In Formula A substitute 3 for m, 2 for n, and $-\frac{1}{2}$ for p, and we obtain

$$\int x^3(a+bx^2)^{-\frac{1}{2}}dx = \frac{x^2(a+bx^2)^{\frac{1}{2}} - 2a\int x(a+bx^2)^{-\frac{1}{2}}dx}{3b}.$$

By a second application of the formula the integral of $x(a+bx^2)^{-\frac{1}{2}}dx$ is found to be $\frac{\sqrt{a+bx^2}}{b}$.

Hence we have

$$\int \frac{x^3 dx}{\sqrt{a+bx^2}} = \left(\frac{x^2}{3b} - \frac{2a}{3b^2}\right)\sqrt{a+bx^2} + C.$$

Ex. 2. Integrate the expression $x^5(a+bx^2)^{-\frac{1}{2}}dx$.

By three successive applications of Formula A we obtain

$$Ans. \left(\frac{x^4}{5b}-\frac{4ax^2}{15b^2}+\frac{8a^2}{15b^3}\right)(a+bx^2)^{\frac{1}{2}}+C.$$

Ex. 3. Integrate the expression $x^{-4}(a+bx^2)^{-\frac{1}{2}}dx$.

In Formula B substitute -4 for m, 2 for n, and $-\frac{1}{2}$ for p, and we obtain

$$\int x^{-4}(a+bx^2)^{-\frac{1}{2}}dx=\frac{x^{-3}(a+bx^2)^{\frac{1}{2}}+2b\int x^{-2}(a+bx^2)^{-\frac{1}{2}}dx}{-3a}.$$

By a second application of the formula the integral of $x^{-2}(a+bx^2)^{-\frac{1}{2}}dx$ is found to be $-\frac{\sqrt{a+bx^2}}{ax}$.

Hence we have $\int\frac{dx}{x^4\sqrt{a+bx^2}}=\left(-\frac{1}{3ax^3}+\frac{2b}{3a^2x}\right)\sqrt{a+bx^2}+C.$

Ex. 4. Integrate the expression $x^{-6}(a+bx^2)^{-\frac{1}{2}}dx$.

By three successive applications of Formula B we obtain

$$Ans. \left(-\frac{1}{5ax^5}+\frac{4b}{15a^2x^3}-\frac{8b^2}{15a^3x}\right)\sqrt{a+bx^2}+C.$$

Ex. 5. Integrate the expression $(a+x^2)^{\frac{1}{2}}dx$.

In Formula C substitute 0 for m, 1 for b, 2 for n, and $\frac{1}{2}$ for p, and we obtain

$$\int(a+x^2)^{\frac{1}{2}}dx=\frac{x(a+x^2)^{\frac{1}{2}}}{2}+\frac{a}{2}\int(a+x^2)^{-\frac{1}{2}}dx.$$

But $$\int\frac{dx}{\sqrt{a+x^2}}=\log.\{x+\sqrt{a+x^2}\}.$$

Hence we have

$$\int(a+x^2)^{\frac{1}{2}}dx=\frac{x(a+x^2)^{\frac{1}{2}}}{2}+\frac{a}{2}\log.\{x+(a+x^2)^{\frac{1}{2}}\}+C.$$

Ex. 6. Integrate the expression $(a+x^2)^{\frac{3}{2}}dx$.

By Formula C,

$$\int(a+x^2)^{\frac{3}{2}}dx=\frac{x(a+x^2)^{\frac{3}{2}}}{4}+\frac{3a}{4}\int(a+x^2)^{\frac{1}{2}}dx.$$

Hence, by Ex. 5,

$$\int(a+x^2)^{\frac{3}{2}}dx=\frac{x(a+x^2)^{\frac{3}{2}}}{4}+\frac{3ax}{8}(a+x^2)^{\frac{1}{2}}+\frac{3a^2}{8}\log.\{x+(a+x^2)^{\frac{1}{2}}\}+C.$$

Ex. 7. Integrate the expression $(a+bx^2)^{-\frac{3}{2}}dx$.
Substituting in Formula D, we obtain

$$\int(a+bx^2)^{-\frac{3}{2}}dx=\frac{x}{a}(a+bx^2)^{-\frac{1}{2}}+\text{C}.$$

Ex. 8. Integrate the expression $(a^2+x^2)^{-2}dx$.
By Formula D,

$$\int(a^2+x^2)^{-2}dx=\frac{x(a^2+x^2)^{-1}}{2a^2}+\frac{1}{2a^2}\int\frac{dx}{a^2+x^2}.$$

But $$\int\frac{dx}{a^2+x^2}=\frac{1}{a}\times\text{the arc whose tangent is }\frac{x}{a}.$$

Hence $$\int(a^2+x^2)^{-2}dx=\frac{x}{2a^2(a^2+x^2)}+\frac{1}{2a^3}\text{ tang.}^{-1}\frac{x}{a}+\text{C}.$$

Ex. 9. Integrate the expression $x^2(2ax-x^2)^{-\frac{1}{2}}dx$.
By two applications of Formula E we obtain

$$Ans.\ -\frac{x+3a}{2}(2ax-x^2)^{\frac{1}{2}}+\frac{3a^2}{2}\text{ ver. sin.}^{-1}\frac{x}{a}+\text{C}.$$

Ex. 10. Integrate the expression $x^{\frac{3}{2}}(2ax-x^2)^{-\frac{1}{2}}dx$.

$$Ans.\ -\frac{8a+2x}{3}(2a-x)^{\frac{1}{2}}+\text{C}.$$

Ex. 11. Integrate the expression $x^4(a+x^2)^{-\frac{1}{2}}dx$.

$$Ans.\left\{\frac{x^3}{4}-\frac{3ax}{8}\right\}\sqrt{a+x^2}+\frac{3a^2}{8}\log.(x+\sqrt{a+x^2})+\text{C}.$$

Ex. 12. Integrate the expression $x^3(a+bx^2)^{\frac{5}{2}}dx$.

$$Ans.\left\{\frac{x^2}{9b}-\frac{2a}{63b^2}\right\}(a+bx^2)^{\frac{7}{2}}+\text{C}.$$

Ex. 13. Integrate the expression $x^7(a+bx^2)^{-\frac{1}{2}}dx$.

$$Ans.\left\{\frac{x^6}{7b}-\frac{6ax^4}{35b^2}+\frac{8a^2x^2}{35b^3}-\frac{16a^3}{35b^4}\right\}(a+bx^2)^{\frac{1}{2}}+\text{C}.$$

Ex. 14. Integrate the expression $x^3(a+bx^2)^{-\frac{3}{2}}dx$.

$$Ans.\left\{\frac{x^2}{b}+\frac{2a}{b^2}\right\}(a+bx^2)^{-\frac{1}{2}}+\text{C}.$$

Ex. 15. Integrate the expression $(a+bx^2)^{-\frac{5}{2}}dx$.

$$Ans. \frac{x}{3a(a+bx^2)^{\frac{3}{2}}}+\frac{2x}{3a^2(a+bx^2)^{\frac{1}{2}}}+C.$$

Ex. 16. Integrate the expression $x^4(1-x^2)^{-\frac{1}{2}}dx$.

$$Ans. -\left\{\frac{x^3}{4}+\frac{3x}{8}\right\}\sqrt{1-x^2}+\frac{3}{8}\sin.^{-1}x+C.$$

Ex. 17. Integrate the expression $x^{-2}(a+bx^2)^{-\frac{3}{2}}dx$.

$$Ans. -\left\{\frac{1}{ax}+\frac{2bx}{a^2}\right\}(a+bx^2)^{-\frac{1}{2}}+C.$$

Ex. 18. Integrate the expression $x^3(a+bx^2)^{-\frac{7}{2}}dx$.

$$Ans. -\left\{\frac{x^2}{3b}+\frac{2a}{15b^2}\right\}(a+bx^2)^{-\frac{5}{2}}+C.$$

Ex. 19. Integrate the expression $x^{-4}(a+bx^2)^{\frac{1}{2}}dx$.

$$Ans. -\frac{(a+bx^2)^{\frac{3}{2}}}{3ax^3}+C.$$

Ex. 20. Integrate the expression $x^3(a+bx^2)^{\frac{7}{2}}dx$.

$$Ans. \left\{\frac{x^2}{11b}-\frac{2a}{99b^2}\right\}(a+bx^2)^{\frac{9}{2}}+C.$$

Either of the Formulas A, B, C, D, and E may become inapplicable on account of the denominator vanishing for special values of m, n, and p; but in such a case the proposed differential expression may be rendered rational, and its integral may be found by methods already explained.

CHAPTER IV.

INTEGRATION BY INFINITE SERIES. INTEGRATION OF LOGARITHMIC, EXPONENTIAL, AND TRIGONOMETRICAL DIFFERENTIALS.

198. When a proposed differential expression can not be integrated by any of the ordinary methods, we may expand it into an infinite series, the terms of which can be integrated separately. If the result is a converging series, the approximate value of the integral may be determined by summing a finite number of terms. This method is also sometimes useful in developing a function into a series.

Ex. 1. Integrate the expression $\frac{dx}{1+x}$, or $(1+x)^{-1}dx$.

By the binomial theorem, we have

$$(1+x)^{-1}=1-x+x^2-x^3+ \text{ etc.}$$

Multiplying by dx, we have

$$(1+x)^{-1}dx=dx-xdx+x^2dx-x^3dx+ \text{ etc.};$$

and integrating each term separately, we obtain

$$\int(1+x)^{-1}dx=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\ldots+\text{C}.$$

But $$\int\frac{dx}{1+x}=\log.(1+x).$$

Hence $$\log.(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+ \text{ etc.}$$

Ex. 2. Integrate the expression $\frac{dx}{1+x^2}$ in an infinite series, and thus obtain a development of tang.^{-1}x.

By the binomial theorem, we have

$$(1+x^2)^{-1}=1-x^2+x^4-x^6+ \text{ etc.}$$

Hence $$(1+x^2)^{-1}dx=dx-x^2dx+x^4dx-x^6dx+ \text{ etc.}$$

Therefore $$\text{tang.}^{-1}x=\int\frac{dx}{1+x^2}=x-\frac{x^3}{3}+\frac{x^5}{5}-\ldots+\text{C}.$$

But when $x=0$, tang.^{-1}x=tang.$^{-1}0=0$. Therefore C$=0$, and

$$\text{tang.}^{-1}x=x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\frac{x^9}{9}-\text{ etc.}$$

Ex. 3. Integrate the expression $\frac{dx}{\sqrt{1-x^2}}$ in an infinite series, and thus obtain a development of sin.^{-1}x.

$$Ans.\ x+\frac{x^3}{2.3}+\frac{1.3x^5}{2.4.5}+\frac{1.3.5x^7}{2.4.6.7}+\text{ etc.}$$

Ex. 4. Integrate the expression $\frac{dx}{a+x}$ in an infinite series, and thus obtain a development of log. $(a+x)$.

$$Ans.\ \log.(a+x)=\log.a+\frac{x}{a}-\frac{x^2}{2a^2}+\frac{x^3}{3a^3}-\frac{x^4}{4a^4}+\text{ etc.}$$

Ex. 5. Integrate the expression $x^{\frac{1}{2}}(1-x^2)^{\frac{1}{2}}dx$.

$$Ans.\ \frac{2x^{\frac{3}{2}}}{3}-\frac{x^{\frac{7}{2}}}{7}-\frac{x^{\frac{11}{2}}}{44}-\frac{x^{\frac{15}{2}}}{120}-\ldots+\text{C.}$$

Ex. 6. Integrate the expression $x^2(1-x^2)^{\frac{1}{2}}dx$.

$$Ans.\ \frac{x^3}{3}-\frac{x^5}{10}-\frac{x^7}{56}-\frac{x^9}{144}-\ldots+\text{C.}$$

INTEGRATION OF LOGARITHMIC DIFFERENTIALS.

199. Logarithmic differentials do not generally admit of exact integrals, but when the exact integral can not be obtained, an approximate integral may be obtained by infinite series. We will examine a few of the simplest forms of logarithmic differentials.

Ex. 1. Integrate the expression $x^m \log. x dx$.

Assume $u=\log. x$, and $dv=x^m dx$;

which give $du=\frac{dx}{x}$, and $v=\frac{x^{m+1}}{m+1}$.

Hence, Art. 187, we obtain

$$\int x^m \log. x dx=\frac{x^{m+1}\log. x}{m+1}-\int\frac{x^m dx}{m+1}$$

$$=\frac{x^{m+1}}{m+1}\left\{\log. x-\frac{1}{m+1}\right\}+\text{C.}$$

Ex. 2. Integrate the expression $x^4 \log. x dx$.

$$Ans. \frac{x^5}{5}\left\{\log. x - \frac{1}{5}\right\} + C.$$

Ex. 3. Integrate the expression $x^m(\log. x)^2 dx$.

Assume $(\log. x)^2 = u$, and $x^m dx = dv$;

which give $du = \frac{2 \log. x dx}{x}$, and $v = \frac{x^{m+1}}{m+1}$.

Hence, Art. 187, we obtain

$$\int x^m(\log. x)^2 dx = \frac{x^{m+1}(\log. x)^2}{m+1} - \frac{2}{m+1}\int x^m \log. x dx.$$

Integrating the last term by Ex. 1, we have

$$\int x^m(\log. x)^2 dx = \frac{x^{m+1}}{m+1}\left\{(\log. x)^2 - \frac{2 \log. x}{m+1} + \frac{2}{(m+1)^2}\right\} + C.$$

Ex. 4. Integrate the expression $x^2(\log. x)^2 dx$.

$$Ans. \frac{x^3}{3}\left\{(\log. x)^2 - \frac{2 \log. x}{3} + \frac{2}{9}\right\} + C.$$

Ex. 5. Integrate the expression $x^m(\log. x)^n dx$.

Assume $(\log. x)^n = u$, and $x^m dx = dv$;

which give $du = n(\log. x)^{n-1}\frac{dx}{x}$, and $v = \frac{x^{m+1}}{m+1}$.

Hence, by Art. 187, we obtain

$$\int x^m(\log. x)^n dx = \frac{x^{m+1}(\log. x)^n}{m+1} - \frac{n}{m+1}\int x^m(\log. x)^{n-1} dx.$$

This formula diminishes the exponent of $\log. x$ by unity, and by successive applications of the formula to the last term of the expression, the exact integral will at length be found by means of Ex. 1, provided n be a positive whole number.

Ex. 6. Integrate the expression $x^5(\log. x)^3 dx$.

$$Ans. \frac{x^6}{6}\left\{(\log. x)^3 - \frac{(\log. x)^2}{2} + \frac{\log. x}{6} - \frac{1}{36}\right\} + C.$$

200. *When m in Ex. 5, Art. 199, is equal to -1.* The formula in Ex. 5 fails when $m = -1$, for the second term of the expression becomes infinite. The integration may, however, be effected as follows:

Assume $\log. x = z.$

Then $\frac{dx}{x} = dz;$

$$\int (\log. x)^n \frac{dx}{x} = \int z^n dz = \frac{z^{n+1}}{n+1} = \frac{(\log. x)^{n+1}}{n+1}.$$

Ex. 1. Integrate the expression $x^{-1} \log. x dx.$

Ans. $\frac{(\log. x)^2}{2} + C.$

Ex. 2. Integrate the expression $x^{-1}(\log. x)^4 dx.$

Ans. $\frac{(\log. x)^5}{5} + C.$

201. *When n in Ex. 5, Art. 199, is negative.* The method given in Ex. 5, Art. 199, fails when the exponent n is negative, since, by application of the formula, the exponent of $\log. x$, instead of approaching zero, continually becomes a greater negative number. In this case we may put the proposed differential under the form

$$x^{m+1}(\log. x)^{-n} \frac{dx}{x}, \text{ or } x^{m+1}(\log. x)^{-n} d \log. x.$$

Assume $u = x^{m+1}$, and $dv = (\log. x)^{-n} d \log. x;$

which give $du = (m+1)x^m dx$, and $v = \frac{(\log. x)^{-n+1}}{-n+1};$

$$\int x^m (\log. x)^{-n} dx = -\frac{x^{m+1}}{(n-1)(\log. x)^{n-1}} + \frac{m+1}{n-1} \int \frac{x^m dx}{(\log. x)^{n-1}}.$$

If n be a whole number, the repeated application of this formula will at length give for the term to be integrated $\frac{x^m dx}{\log. x}$.

In order to simplify this expression, put $z = x^{m+1}$.

Whence $(m+1) \log. x = \log. z$, or $\log. x = \frac{\log. z}{m+1}$.

Also, by differentiating,

$$(m+1)x^m dx = dz.$$

Whence $x^m dx = \frac{dz}{m+1}.$

Therefore $\frac{x^m dx}{\log. x} = \frac{dz}{\log. z},$

a differential which can only be integrated by series.

Ex. 1. Integrate the expression $x^4(\log. x)^{-2}dx$.

By the preceding formula,

$$\int \frac{x^4 dx}{(\log. x)^2} = -\frac{x^5}{\log. x} + 5\int \frac{x^4 dx}{\log. x}.$$

Ex. 2. Integrate the expression $\frac{x^4 dx}{(\log. x)^3}$.

$$Ans. -\frac{x^5}{2\,(\log. x)^2} - \frac{5x^5}{2\,\log. x} + \frac{25}{2}\int \frac{x^4 dx}{\log. x}.$$

202. By similar methods we may integrate $X(\log. x)^n dx$, in which X denotes any rational function of x.

Ex. 1. Integrate the expression $\frac{\log. x dx}{(1-x)^2}$.

Assume $u = \log. x$, and $dv = (1-x)^{-2}dx$;

which give $du = \frac{dx}{x}$, and $v = (1-x)^{-1}$.

Hence, Art. 187, we obtain

$$\int \frac{\log. x dx}{(1-x)^2} = \frac{\log. x}{1-x} - \int \frac{dx}{x(1-x)}$$

$$= \frac{\log. x}{1-x} - \int \frac{dx}{x} - \int \frac{dx}{1-x}.$$

$$= \frac{x \log. x}{1-x} + \log. (1-x) + C.$$

Ex. 2. Integrate the expression $(a+bx)^m \log. x dx$.

$$Ans. \frac{(a+bx)^{m+1} \log. x}{(m+1)b} - \frac{1}{(m+1)b}\int \frac{(a+bx)^{m+1} dx}{x}.$$

INTEGRATION OF EXPONENTIAL DIFFERENTIALS.

203. Exponential differentials are closely related to logarithmic differentials, and do not generally admit of exact integrals; but we can usually cause the value of the required integral to depend upon another which is of a simpler form. We will consider a few of the simplest cases.

Ex. 1. Integrate the expression $xa^x dx$.

Assume $u=x$, and $dv=a^x dx$;

which give $du=dx$, and $v=\frac{a^x}{\log. a}$ (see Art. 40).

Hence $$\int xa^x dx=\frac{xa^x}{\log. a}-\int\frac{a^x dx}{\log. a}$$
$$=\frac{xa^x}{\log. a}-\frac{a^x}{(\log. a)^2}+\mathrm{C}.$$

Ex. 2. Integrate the expression $x^m a^x dx$.

Assume $u=x^m$, and $dv=a^x dx$;

which give $du=mx^{m-1}dx$, and $v=\frac{a^x}{\log. a}$.

Hence $$\int x^m a^x dx=\frac{x^m a^x}{\log. a}-\frac{m}{\log. a}\int x^{m-1}a^x dx.$$

By successive applications of this formula the exponent of x, if positive, will be continually diminished, and if it be also a whole number, the exact integral may be determined. If the exponent of x be a fraction, the integral can only be found in an infinite series.

Ex. 3. Integrate the expression $x^3 a^x dx$.

$$Ans.\ a^x\left\{\frac{x^3}{\log. a}-\frac{3x^2}{(\log. a)^2}+\frac{6x}{(\log. a)^3}-\frac{6}{(\log. a)^4}\right\}.$$

204. *When the exponent m in Ex. 2, Art. 203, is negative.* The method given in Ex. 2, Art. 203, fails when the exponent m is negative, since, by successive applications of the formula, the exponent of x, instead of approaching zero, is increased numerically. In this case we may proceed as follows:

Assume $u=a^x$, and $dv=x^{-m}dx$;

which give $du=a^x \log. a dx$, and $v=-\frac{1}{(m-1)x^{m-1}}$.

Hence $$\int\frac{a^x dx}{x^m}=-\frac{a^x}{(m-1)x^{m-1}}+\frac{\log. a}{m-1}\int\frac{a^x dx}{x^{m-1}}.$$

Ex. Integrate the expression $\frac{a^x dx}{x^3}$.

$$Ans.\ -\frac{a^x}{2x^2}-\frac{a^x \log. a}{2x}+\frac{(\log. a)^2}{2}\int\frac{a^x dx}{x}.$$

The expression $\frac{a^x dx}{x}$ can only be integrated by development into series, as shown in the following Article.

205. *When the exponent m in Art. 204 is equal to unity.* The formula in Art. 204 fails when the exponent m is equal to unity, since the denominator of the second member becomes zero. In this case the integral can only be determined by developing a^x into an infinite series, and integrating each term separately. According to Ex. 12, Art. 56, we have

$$a^x = 1 + x \log. a + \frac{x^2}{2}(\log. a)^2 + \frac{x^3}{2.3}(\log. a)^3 + \text{ etc.}$$

Hence $$\frac{a^x dx}{x} = \frac{dx}{x} + \log. a dx + \frac{x}{2}(\log. a)^2 dx + \frac{x^2}{2.3}(\log. a)^3 dx + \text{ etc.}$$

Therefore

$$\int \frac{a^x dx}{x} = \log. x + x \log. a + \frac{x^2}{4}(\log. a)^2 + \frac{x^3}{18}(\log. a)^3 + \text{ etc.}$$

This formula solves the exceptional case mentioned in Art. 204, and shows the intimate relation between logarithmic and exponential integrals.

206. By a similar method we may integrate the expression $Xa^x dx$, in which X represents any algebraic function of x.

Assume $u = X$, and $dv = a^x dx$.

Now $da^x = a^x \log. a dx$;

that is, $$a^x dx = \frac{da^x}{\log. a},$$

and $$\int a^x dx = \frac{a^x}{\log. a} = v.$$

Hence $$\int Xa^x dx = \frac{Xa^x}{\log. a} - \frac{1}{\log. a}\int a^x d\,X. \qquad (1)$$

By the application of this formula the integral of $Xa^x dx$ is made to depend upon the integral of $a^x d\,X$. If the function X is of such a form that one of its differential coefficients is constant, the next differential coefficient will be zero. In this case the exact integral may be obtained by successive applications of formula (1).

207. *When the constant* (a) *in Art.* 203 *is the base of the system of logarithms.* If we make the constant a in Art. 203 equal to the base of the Naperian system of logarithms, then, since the logarithm of the base is unity, the preceding results will assume a simpler form. In the following examples the base of the Naperian system is represented by e.

Ex. 1. Integrate the expression $xe^x dx$. *Ans.* $e^x(x-1)+C$.

Ex. 2. Integrate the expression $x^2e^x dx$.

Ans. $e^x(x^2-2x+2)+C$.

Ex. 3. Integrate the expression $x^3e^x dx$.

Ans. $e^x(x^3-3x^2+6x-6)+C$.

Ex. 4. Integrate the expression $x^4e^x dx$.

Ans. $e^x(x^4-4x^3+12x^2-24x+24)+C$.

Ex. 5. Integrate the expression $\frac{e^x dx}{x}$.

Ans. $\log. x+x+\frac{x^2}{2^2}+\frac{x^3}{2.3^2}+$ etc.

Ex. 6. Integrate the expression $\frac{e^x dx}{x^2}$.

Ans. $-\frac{e^x}{x}+\log. x+x+\frac{x^2}{2^2}+\frac{x^3}{2.3^2}+$ etc.

Ex. 7. Integrate the expression $\frac{xdx}{e^x}$.

Assume $u=x$, and $dv=e^{-x}dx$;

which give $du=dx$, and $v=-e^{-x}$.

Hence $$\int\frac{xdx}{e^x}=-e^{-x}.x+\int\frac{dx}{e^x}$$
$$=-e^{-x}.x-e^{-x}+C.$$

Ex. 8. Integrate the expression $\frac{x^2dx}{e^x}$.

Assume $u=x^2$, and $dv=e^{-x}dx$;

which give $du=2xdx$, and $v=-e^{-x}$.

Hence $$\int\frac{x^2dx}{e^x}=-e^{-x}.x^2+\int\frac{2xdx}{e^x}$$
$$=-e^{-x}(x^2+2x+2)+C.$$

INTEGRATION OF TRIGONOMETRICAL DIFFERENTIALS.

208. Trigonometrical differentials may sometimes be reduced to algebraic forms by the substitution of a new variable, and, when this is impossible, they may sometimes be integrated by methods similar to those employed in the preceding Articles.

Ex. Integrate the expression $(\sin. x)^m dx$.

Assume $\sin. x = z.$

Whence $\cos. x = (1-z^2)^{\frac{1}{2}};$

$$dx = \frac{dz}{\cos. x} = (1-z^2)^{-\frac{1}{2}} dz.$$

Hence $$\int (\sin. x)^m dx = \int z^m (1-z^2)^{-\frac{1}{2}} dz.$$

This expression can be integrated by the method of Arts. 192 and 193. When m is a whole number, by repeated applications of Formula A or B the integral may be made to depend upon $\int \frac{dz}{(1-z^2)^{\frac{1}{2}}}$, which is equal to x, or upon $\int \frac{zdz}{(1-z^2)^{\frac{1}{2}}}$, which is equal to $-\cos. x$.

In particular cases simpler methods of integration are possible.

209. *When the exponent m in Art.* 208 *is an odd positive integer.*

Suppose $$m = 2r+1.$$

Since $$(\sin. x)^m = (\sin.^2 x)^r \sin. x = (1-\cos.^2 x)^r \sin. x,$$

we have $$\int (\sin. x)^m dx = \int (1-\cos.^2 x)^r \sin. x dx$$
$$= -\int (1-\cos.^2 x)^r d \cos. x.$$

Since r is an integer, this binomial can be expanded in a finite number of terms, and each term can then be integrated separately. If m is an odd negative integer, this substitution gives a rational fraction in $\cos. x$ which is integrable.

Ex. 1. Integrate the expression $(\sin. x)^3 dx$.

$$Ans. \quad -\int (1-\cos.^2 x) d \cos. x = -\cos. x + \frac{\cos.^3 x}{3}.$$

Ex. 2. Integrate the expression $(\sin. x)^5 dx$.

$$Ans. \quad -\int (1-\cos.^2 x)^2 d \cos. x = -\cos. x + \frac{2 \cos.^3 x}{3} - \frac{\cos.^5 x}{5}.$$

210. *When the exponent m in Art. 208 is an even positive integer.* In this case we may develop the power of $\sin. x$ in a series of terms of sines or cosines of multiples of the arc x, and then integrate each term separately.

By Trig., Art. 74, $2(\sin. x)^2 = 1 - \cos. 2x.$

By involution and substitution, we obtain

$$8(\sin. x)^4 = 3 - 4\cos. 2x + \cos. 4x;$$

$$32(\sin. x)^6 = 10 - 15\cos. 2x + 6\cos. 4x - \cos. 6x.$$

Ex. 1. Integrate the expression $(\sin. x)^2 dx$.

$$Ans. \; -\tfrac{1}{2}\int \cos. 2x dx + \int \frac{dx}{2} = -\frac{\sin. 2x}{4} + \frac{x}{2} + C.$$

Ex. 2. Integrate the expression $(\sin. x)^4 dx$.

$$Ans. \; \frac{\sin. 4x}{32} - \frac{\sin. 2x}{4} + \frac{3x}{8} + C.$$

Ex. 3. Integrate the expression $(\sin. x)^6 dx$.

$$Ans. \; -\frac{\sin. 6x}{192} + \frac{3\sin. 4x}{64} - \frac{15\sin. 2x}{64} + \frac{5x}{16} + C.$$

211. It is required to integrate the expression $(\cos. x)^n dx$.

Assume $\cos. x = z.$

Whence $\sin. x = (1 - z^2)^{\frac{1}{2}}.$

Hence $\int (\cos. x)^n dx = -\int z^n (1 - z^2)^{-\frac{1}{2}} dz,$

which expression may be integrated as in Art. 208.

When n is an odd positive or negative integer, we may proceed as in Art. 209, and will have

$$\int (\cos. x)^n dx = \int (1 - \sin.^2 x)^r d \sin. x,$$

where the binomial may be expanded, and each term integrated separately.

Ex. 1. Integrate the expression $(\cos. x)^3 dx$.

$$Ans. \; \sin. x - \frac{(\sin. x)^3}{3} + C.$$

Ex. 2. Integrate the expression $(\cos. x)^5 dx$.

$$Ans. \; \sin. x - \frac{2(\sin. x)^3}{3} + \frac{(\sin. x)^5}{5} + C.$$

When n is an even positive integer, we may develop the

power of cos. x in a series of terms of cosines of multiples of the arc x, and then integrate each term separately.

By Trig., Art. 74, $2(\cos. x)^2 = 1 + \cos. 2x$.

By involution and substitution, we obtain

$$8(\cos. x)^4 = 3 + 4\cos. 2x + \cos. 4x;$$

$$32(\cos. x)^6 = 10 + 15\cos. 2x + 6\cos. 4x + \cos. 6x.$$

Ex. 3. Integrate the expression $(\cos. x)^2 dx$.

$$Ans. \frac{x}{2} + \frac{\sin. 2x}{4}.$$

Ex. 4. Integrate the expression $(\cos. x)^4 dx$.

$$Ans. \frac{3x}{8} + \frac{\sin. 2x}{4} + \frac{\sin. 4x}{32}.$$

Ex. 5. Integrate the expression $(\cos. x)^6 dx$.

$$Ans. \frac{5x}{16} + \frac{15\sin. 2x}{64} + \frac{3\sin. 4x}{64} + \frac{\sin. 6x}{192}.$$

212. It is required to integrate the expression

$$(\sin. x)^m (\cos. x)^n dx.$$

Assume $\sin. x = z$.

Whence $\cos. x = (1 - z^2)^{\frac{1}{2}}$;

$$dx = (1 - z^2)^{-\frac{1}{2}} dz.$$

Hence $\int (\sin. x)^m (\cos. x)^n dx = \int z^m (1 - z^2)^{\frac{n}{2}} (1 - z^2)^{-\frac{1}{2}} dz$

$$= \int z^m (1 - z^2)^{\frac{n-1}{2}} dz,$$

which may be reduced by Formulas A, B, C, and D, and in some cases integrated.

When either m or n is odd and integral, whether positive or negative, it is more convenient to follow the method of Art. 209. If $m = 2r + 1$, we shall have

$$\int (\sin. x)^m (\cos. x)^r dx = -\int (\cos. x)^n (1 - \cos.^2 x)^r d\cos. x,$$

which may be expanded and integrated as in Art. 209.

If $n = 2r + 1$, we shall have

$$\int (\sin. x)^m (\cos. x)^n dx = \int (\sin. x)^m (1 - \sin.^2 x)^r d\sin. x.$$

If both m and n are positive and even integers, substitute $2\sin.^2 x = 1 - \cos. 2x$ and $2\cos.^2 x = 1 + \cos. 2x$, expand and reduce by Art. 211.

Ex. 1. Integrate the expression $\sin.^3x \cos.^2x dx$.

$$Ans. \frac{(\cos. x)^5}{5} - \frac{(\cos. x)^3}{3}.$$

Ex. 2. Integrate the expression $(\sin. x)^5(\cos. x)^4 dx$.

$$Ans. -\frac{(\cos. x)^5}{5} + \frac{2(\cos. x)^7}{7} - \frac{(\cos. x)^9}{9}.$$

Ex. 3. Integrate the expression $(\sin. x)^4(\cos. x)^3 dx$.

$$Ans. \frac{(\sin. x)^5}{5} - \frac{(\sin. x)^7}{7}.$$

Ex. 4. Integrate the expression $(\sin. x)^6(\cos. x)^3 dx$.

$$Ans. \frac{(\sin. x)^7}{7} - \frac{(\sin. x)^9}{9}.$$

213. It is required to integrate the expression $(\text{tang. } x)^m dx$.

Assume $\text{tang. } x = z.$

Then $$dx = \frac{dz}{1+z^2},$$

and $$\int (\text{tang. } x)^m dx = \int \frac{z^m dz}{1+z^2},$$

which is a rational fraction, and may be integrated by the method of Art. 173.

Ex. 1. Integrate the expression $(\text{tang. } x)^4 dx$.

$$(\text{tang. } x)^4 dx = \frac{z^4 dz}{1+z^2} = z^2 dz - dz + \frac{dz}{1+z^2}.$$

Hence $$\int (\text{tang. } x)^4 dx = \frac{(\text{tang. } x)^3}{3} - \text{tang. } x + x.$$

Ex. 2. Integrate the expression $(\text{tang. } x)^5 dx$.

$$Ans. \frac{(\text{tang. } x)^4}{4} - \frac{(\text{tang. } x)^2}{2} + \log. \sec. x.$$

214. It is required to integrate the expression $x^n \sin. x dx$.

This expression may be integrated by the method of Art. 187.

Assume $u = x^n$, and $dv = \sin. x dx$;

which give $du = nx^{n-1} dx$, and $v = -\cos. x$.

Then $$\int x^n \sin. x dx = -x^n \cos. x + n \int x^{n-1} \cos. x dx. \qquad (1)$$

To integrate the second term in this equation, assume

$$u = x^{n-1}, \text{ and } dv = \cos. x dx;$$

which give $du = (n-1)x^{n-2}dx$, and $v = \sin. x$.

Then $\int x^{n-1} \cos. x dx = x^{n-1} \sin. x - (n-1)\int x^{n-2} \sin. x dx.$ (2)

The second term of this expression may be integrated by formula (1), and we shall obtain the development

$$\int x^n \sin. x dx = -x^n \cos. x + nx^{n-1} \sin. x + n(n-1)x^{n-2} \cos. x - n(n-1)(n-2)x^{n-3} \sin. x - \text{ etc. (two plus and two minus terms alternately).}$$

Ex. 1. Integrate the expression $x \sin. x dx$.

Ans. $-x \cos. x + \sin. x$.

Ex. 2. Integrate the expression $x^2 \sin. x dx$.

Ans. $-x^2 \cos. x + 2x \sin. x + 2 \cos. x$.

Ex. 3. Integrate the expression $x^3 \sin. x dx$.

Ans. $-x^3 \cos. x + 3x^2 \sin. x + 6x \cos. x - 6 \sin. x$.

Ex. 4. Integrate the expression $x^4 \sin. x dx$.

Ans. $-x^4 \cos. x + 4x^3 \sin. x + 12x^2 \cos. x - 24x \sin. x - 24 \cos. x$.

215. It is required to integrate the expression $x^n \cos. x dx$. Proceeding as in the last Article, we shall find

$$\int x^n \cos. x dx = x^n \sin. x + nx^{n-1} \cos. x - n(n-1)x^{n-2} \sin. x - n(n-1)(n-2)x^{n-3} \cos. x + \text{ etc. (two plus and two minus terms alternately).}$$

Ex. 1. Integrate the expression $x \cos. x dx$.

Ans. $x \sin. x + \cos. x$.

Ex. 2. Integrate the expression $x^2 \cos. x dx$.

Ans. $x^2 \sin. x + 2x \cos. x - 2 \sin. x$.

Ex. 3. Integrate the expression $x^3 \cos. x dx$.

Ans. $x^3 \sin. x + 3x^2 \cos. x - 6x \sin. x - 6 \cos. x$.

216. *Inverse trigonometrical differentials.* Differential expressions containing x, dx, and $\sin.^{-1}x$, or $\cos.^{-1}x$, or $\tang.^{-1}x$, etc., may frequently be integrated by methods similar to those already explained.

Ex. 1. Integrate the expression $x^2 \sin.^{-1}x dx$.

Integrating by parts (Art. 187), place

$$x^2dx = dv, \text{ and } \sin.^{-1}x = u.$$

Then $$v = \int x^2 dx = \frac{x^3}{3}, \text{ and } du = \frac{dx}{\sqrt{1-x^2}}.$$

Substituting in the formula

$$\int udv = uv - \int vdu,$$

we have $$\int x^2 \sin.^{-1}x dx = \frac{x^3}{3} \sin.^{-1}x - \int \frac{x^3 dx}{3\sqrt{1-x^2}}.$$

Integrating by Formula A, Art. 192, we have

$$\int \frac{x^3 dx}{\sqrt{1-x^2}} = -\left(\frac{x^2}{3} + \frac{2}{3}\right)\sqrt{1-x^2}.$$

Hence $$\int x^2 \sin.^{-1}x dx = \frac{x^3}{3} \sin.^{-1}x + \frac{1}{3}\left(\frac{x^2}{3} + \frac{2}{3}\right)\sqrt{1-x^2} + C.$$

Ex. 2. Integrate the expression $x^4 \tang.^{-1}x dx$.

Place $$x^4dx = dv, \text{ and } \tang.^{-1}x = u.$$

Then $$v = \frac{x^5}{5}, \text{ and } du = \frac{dx}{1+x^2}.$$

Hence

$$\int x^4 \tang.^{-1}x dx = \frac{x^5}{5} \tang.^{-1}x - \int \frac{x^5 dx}{5(1+x^2)}$$

$$= \frac{x^5}{5} \tang.^{-1}x - \frac{1}{5}\left\{\frac{x^4}{4} - \frac{x^2}{2} + \frac{1}{2}\log.(1+x^2)\right\} + C.$$

Ex. 3. Integrate the expression $x \tang.^{-1}x dx$.

$$Ans. \frac{1+x^2}{2} \tang.^{-1}x - \frac{x}{2} + C.$$

CHAPTER V.

SUCCESSIVE INTEGRATION. INTEGRATION OF FUNCTIONS OF TWO OR MORE VARIABLES. DIFFERENTIAL EQUATIONS.

217. *Successive integration.* In the examples of the preceding Chapters the first differential coefficient has been given to determine the primitive function from which it was derived. When, however, instead of the first differential coefficient, the second differential coefficient is given, then, by a first integration we shall arrive at the first differential coefficient, and it is only after a second integration that we arrive at the original function. If the third differential coefficient were given, it would require three successive integrations to arrive at the original function, and so on. Since each integration introduces a constant, it follows that the complete primitive ought to contain as many arbitrary constants as it has required integrations to obtain it.

A double integration is sometimes indicated by the symbol $\int\int$, and sometimes by the symbol $\int^2$. So also a triple integral may be indicated by $\int\int\int$, or by $\int^3$, and so on.

Ex. 1. Integrate the expression $d^2u = ax^2dx^2$ (see Art. 54).

This expression may be written

$$\frac{d^2u}{dx} = ax^2dx.$$

Hence
$$\int\frac{d^2u}{dx} = \int ax^2dx;$$

or
$$\frac{du}{dx} = \frac{ax^3}{3} + C.$$

Hence
$$du = \frac{ax^3dx}{3} + Cdx.$$

Integrating again, we obtain

$$u=\frac{ax^4}{12}+Cx+C';$$

that is,
$$u=\iint ax^2dx^2=\frac{ax^4}{12}+Cx+C'.$$

Ex. 2. Integrate the expression $d^3u=axdx^3$.

This expression may be written

$$\frac{d^3u}{dx^2}=d\left(\frac{d^2u}{dx^2}\right)=axdx.$$

Hence
$$\frac{d^2u}{dx^2}=\frac{ax^2}{2}+C;$$

and finally, as in the former example,

$$u=\int^3 axdx^3=\frac{ax^4}{24}+\frac{Cx^2}{2}+C'x+C''.$$

Ex. 3. Integrate the expression $d^4u=144dx^4$.

Proceeding as in the last example, we find

$$\frac{d^3u}{dx^3}=144x+C;$$

$$\frac{d^2u}{dx^2}=72x^2+Cx+C';$$

$$\frac{du}{dx}=24x^3+\frac{Cx^2}{2}+C'x+C'';$$

$$u=\int^4 144dx^4=6x^4+\frac{Cx^3}{6}+\frac{C'x^2}{2}+C''x+C'''$$

Ex. 4. Integrate the expression $d^3u=ax^{\frac{1}{2}}dx^3$.

$$Ans.\ u=\frac{8ax^{\frac{7}{2}}}{105}+\frac{Cx^2}{2}+C'x+C''.$$

Ex. 5. Integrate the expression $d^4u=-x^{-4}dx^4$.

$$Ans.\ u=\frac{\log. x}{6}+\frac{Cx^3}{6}+\frac{C'x^2}{2}+C''x+C'''.$$

218. *Integration of differential functions of two variables.* If u is a function of x and y where each variable is supposed to be entirely independent of the other, we have found, Art. 66,

$$Du=\frac{du}{dx}dx+\frac{du}{dy}dy,$$

where Du denotes the *total* differential of the function, while the first du in the second member of the equation denotes the partial differential taken with respect to x, and the second du denotes the partial differential taken with respect to y.

If, then, we have an expression of the form

$$du=Mdx+Ndy, \qquad (1)$$

where M and N are functions of x and y, if du be the total differential of the function, we must have

$$\frac{du}{dx}=M, \text{ and } \frac{du}{dy}=N.$$

Differentiating the first expression with respect to y, and the second with respect to x, we have

$$\frac{dM}{dy}=\frac{d^2u}{dxdy}, \text{ and } \frac{dN}{dx}=\frac{d^2u}{dydx}.$$

By a method similar to that employed in Art. 62, it may be proved that

$$\frac{d^2u}{dxdy}=\frac{d^2u}{dydx},$$

and hence

$$\frac{dM}{dy}=\frac{dN}{dx}; \qquad (2)$$

or, in order that $Mdx+Ndy$ shall be a total differential, it is necessary that

$$\frac{dM}{dy}=\frac{dN}{dx},$$

which is hence called the *criterion of integrability.*

When an expression of the form (1) is to be integrated, we first examine whether the criterion (2) is satisfied. If this criterion is satisfied, the equation is integrable. To perform the integration, we integrate one of the partial differentials (as Mdx) with reference to its corresponding variable, as though the other variable were constant, and add such a function of y as will satisfy the equation $\frac{du}{dy}=N$.

Integrating on the supposition that y is constant, we have

$$u = \int M dx + C,$$

where the correction C is a function of y alone.

Ex. 1. Integrate the expression $du = y dx + x dy$.

Here $M = y$, and $N = x$;

$$\frac{dM}{dy} = 1 = \frac{dN}{dx}.$$

Hence criterion (2) is satisfied.

$$u = \int M dx = xy + C.$$

Ex. 2. Integrate the expression $du = 2xy^3 dx + 3x^2y^2 dy$.

Here $M = 2xy^3$, and $N = 3x^2y^2$;

$$\frac{dM}{dy} = 6xy^2 = \frac{dN}{dx}.$$

Criterion (2) is satisfied.

$$u = \int M dx = x^2y^3 + C.$$

Ex. 3. Integrate the expression

$$du = (ay^2 - by^3)dx + (2axy - 3by^2x)dy.$$

Here $M = ay^2 - by^3$, and $N = 2axy - 3by^2x$;

$$\frac{dM}{dy} = 2ay - 3by^2 = \frac{dN}{dx}.$$

Criterion (2) is satisfied.

$$u = \int (ay^2 - by^3)dx = ay^2x - by^3x + C.$$

Ex. 4. Integrate the expression

$$du = (2y^2x + 9x^2y + 8x^3)dx + (2x^2y + 3x^3)dy.$$

Ans. $u = y^2x^2 + 3x^3y + 2x^4 + C.$

Ex. 5. Integrate the expression

$$du = 2axy dx + ax^2 dy - y^3 dx - 3xy^2 dy.$$

Ans. $u = ax^2y - y^3x + C.$

219. *Differential equations.* A differential equation is an equation involving differential coefficients with or without the primitive variables from which those differential coefficients were derived. Thus in Arts. 68–70 we have found such differential equations as the following:

$$dy = mdx;$$
$$ydy = 2adx;$$
$$2ydy = mdx + 2nxdx;$$
$$dy^2 + yd^2y = ndx^2.$$

220. *Classification of differential equations.* Differential equations are classified according to their *order and degree.*

The *order* of a differential equation is the same as the order of the highest differential coefficient which it contains.

The *degree* of a differential equation is the same as the degree to which the differential coefficient which marks its order is raised, that coefficient being supposed to enter into the equation in a rational form.

Thus the equation $\frac{dy}{dx} = \frac{2a}{y}$ is an equation of the first order and the first degree.

The equation $\left(\frac{dy}{dx}\right)^2 + a\left(\frac{dy}{dx}\right) = b$ is of the first order and of the second degree.

The equation $\frac{d^2y}{dx^2} + a\left(\frac{dy}{dx}\right) + by^2 = c$ is of the second order and of the first degree.

The equation $\left(\frac{d^2y}{dx^2}\right)^2 + a\left(\frac{d^2y}{dx^2}\right) = b$ is of the second order and second degree.

221. *Differential equations of the first order and degree between two variables.*

Differential equations of the first order and degree between two variables may be represented under the general form

$$M + \frac{Ndy}{dx} = 0,$$

M and N being functions of the variables x and y.

This equation is commonly written

$$Mdx + Ndy = 0. \qquad (1)$$

There are certain cases in which equation (1) admits of solution.

222. Case I. *When the variables in* M *and* N *can be separated.* Equation (1) can be solved when the variables in M and N admit of being separated; that is, when the equation can be reduced to the form

$$Xdx+Ydy=0, \qquad (2)$$

in which X is a function of x alone, and Y a function of y alone.

To solve the equation in its reduced form (2), it is only necessary to integrate the two terms separately, and to equate the result to an arbitrary constant. Thus the solution will be

$$\int Xdx+\int Ydy=C.$$

Ex. 1. Solve the equation $xdx+ydy=0$.

By integration, we obtain

$$\frac{x^2}{2}+\frac{y^2}{2}=C;$$

or, since C is arbitrary, we may write it

$$x^2+y^2=C.$$

Ex. 2. Solve the equation $ydx-xdy=0$.

Dividing by xy, we have

$$\frac{dx}{x}=\frac{dy}{y}.$$

Integrating, we have $\log. x=\log. y+C$;

or we may write it

$$\log. x=\log. y+\log. a=\log. ay.$$

Hence

$$x=ay.$$

Ex. 3. Solve the equation $(1+x^2)dy=y^{\frac{1}{2}}dx$.

Here

$$\frac{dy}{y^{\frac{1}{2}}}=\frac{dx}{1+x^2}.$$

Integrating, we have

$$2y^{\frac{1}{2}}=\text{tang.}^{-1}x+C.$$

Ex. 4. Solve the equation $xy^2dx+dy=0$.

Dividing by y^2, $$xdx+\frac{dy}{y^2}=0.$$

Integrating, we have $$\frac{x^2}{2}-\frac{1}{y}=C;$$

or $$x^2y-2=2Cy.$$

Ex. 5. Solve the equation $\frac{dx}{1+x}+\frac{dy}{1+y}=0$.

Ans. $(1+x)(1+y)=C$.

223. Case II. When one side of the equation may be rendered an exact differential, and the other a function which contains only one of the variables.

Ex. 1. Solve the equation $\frac{xdy+ydx}{x}=-dy$.

Dividing by y, we have

$$\frac{xdy+ydx}{xy}=-\frac{dy}{y}.$$

Integrating, we have $$\log.(xy)=\log.\frac{1}{y}+C.$$

Hence $$xy=\frac{a}{y},$$

or $$xy^2=a.$$

Ex. 2. Solve the equation $ydx=y^3dy+y^2dy+xdy$.

Transposing, we obtain

$$\frac{ydx-xdy}{y^2}=ydy+dy.$$

Integrating, we have $$\frac{x}{y}=\frac{y^2}{2}+y+C.$$

Ex. 3. Solve the equation $ydy-xdx=a(xdy+ydx)$.

Multiplying by 2, we have

$$d(y^2-x^2)=2ad(xy).$$

Hence $$y^2-x^2=2axy+C.$$

Ex. 4. Solve the equation $adx^2=ydy^2-bdxdy$.

This equation may be reduced by completing the square, as in an algebraic equation of the second degree, and we have

$$dx^2+\frac{b}{a}dydx+\frac{b^2dy^2}{4a^2}=\frac{4ay+b^2}{4a^2}dy^2.$$

Hence $$dx+\frac{bdy}{2a}=\frac{(4ay+b^2)^{\frac{1}{2}}}{2a}dy.$$

Therefore $$x+\frac{by}{2a}=\frac{(4ay+b^2)^{\frac{3}{2}}}{12a^2}+\mathrm{C}.$$

224. Case III. *When the equation is homogeneous.* The differential equation $\mathrm{M}dx+\mathrm{N}dy=0$ is said to be homogeneous when M and N are homogeneous functions of x and y, and are of the same degree. To integrate a homogeneous equation, it suffices to assume $x=vy$, or $y=vx$. In the transformed equation the variables x and v will then admit of separation.

Ex. 1. Solve the equation $y^2dy=3yxdx-x^2dy$.

Substitute vy for x, and we have

$$y^2dy=3y^2v(vdy+ydv)-y^2v^2dy.$$

Dividing by y^2,

$$dy=3v(ydv+vdy)-v^2dy=3yvdv+2v^2dy.$$

Hence $$\frac{dy}{y}=\frac{3vdv}{1-2v^2}.$$

Therefore $$\log. y=\tfrac{3}{4}\log.\frac{c}{1-2v^2};$$

$$y^4=\frac{c^3}{(1-2v^2)^3}=\frac{c^3y^6}{(y^2-2x^2)^3},$$

or $$y^2-2x^2=cy^{\frac{2}{3}}.$$

Ex. 2. Solve the equation $y^2dx+xydy+x^2dy=0$.

Put $$x=vy.$$

Then $$y^2vdy+y^3dv+y^2vdy+y^2v^2dy=0,$$

or $$2vdy+v^2dy+ydv=0,$$

or $$\frac{dy}{y}=-\frac{dv}{2v+v^2}=\frac{dv}{2(2+v)}-\frac{dv}{2v}.$$

Hence

$$\log. y=\tfrac{1}{2}\log.(2+v)-\tfrac{1}{2}\log. v=\tfrac{1}{2}\log.\frac{2+v}{v}=\log.\sqrt{\frac{2+v}{v}}+\mathrm{C}.$$

Therefore

$$y=a\left(\frac{2+v}{v}\right)^{\frac{1}{2}};$$

$$y^2=\frac{2+v}{v}.a^2=\frac{2y+x}{x}.a^2;$$

$$y^2x=(2y+x)a^2.$$

Ex. 3. Solve the equation $x^2dy-y^2dx=xydx$.

Put $$y=vx.$$

Then $$x^2vdx+x^3dv-v^2x^2dx=x^2vdx.$$

Hence $$xdv=v^2dx,$$

or $$\frac{dx}{x}=\frac{dv}{v^2}.$$

Hence $$\log. x=-\frac{1}{v}+c=-\frac{x}{y}+\mathrm{C}.$$

225. *Geometrical applications of differential equations.* By means of the preceding principles we are often able to deduce the equation of a curve which possesses some remarkable property.

Ex. 1. Determine the curve whose subnormal is constant.

The general expression for the subnormal (Art. 93) is $\frac{ydy}{dx}$. Denoting the constant by a, we have the equation

$$\frac{ydy}{dx}=a.$$

Whence $$ydy=adx;$$

and integrating, $$\frac{y^2}{2}=ax+\mathrm{C},$$

or $$y^2=2ax+2\mathrm{C},$$

which is the equation of a parabola.

Ex. 2. Determine the curve whose subtangent is constant.

The general expression for the subtangent (Art. 91) is $\frac{ydx}{dy}$.

Hence $$\frac{ydx}{dy}=a,$$

or $$dx=\frac{ady}{y}.$$

Therefore $$x=a.\log.y+C,$$

which is the equation of the logarithmic curve, the modulus of the system being a.

Ex. 3. Determine the curve whose subtangent is double the corresponding abscissa.

Here we have $$\frac{ydx}{dy}=2x.$$

Whence $$\frac{2dy}{y}=\frac{dx}{x},$$

or $$2\log.y=\log.x+C,$$

or $$y^2=ax,$$

which is the equation of a common parabola.

Ex. 4. Determine the curve whose subtangent is three times the corresponding abscissa.

$$\frac{ydx}{dy}=3x;$$

$$\frac{3dy}{y}=\frac{dx}{x};$$

$$3\log.y=\log.x+C,$$

or $$y^3=ax.$$

Ex. 5. Determine the curve whose subtangent is n times the corresponding abscissa. *Ans.* $y^n=ax$.

This Chapter exhibits a few of the simplest properties of differential equations. The student who desires to pursue the subject further is referred to larger treatises, such as those of Young, Todhunter, Jephson, De Morgan, Price, Boole, and others.

CHAPTER VI.

QUADRATURE OF PLANE SURFACES. RECTIFICATION OF PLANE CURVES. SURFACES AND VOLUMES OF SOLIDS OF REVOLUTION.

Quadrature of Plane Surfaces.

226. *The quadrature of a surface* is finding the measure of its area, or finding a square equivalent to a proposed area. Thus the quadrature of the circle consists in finding a square which contains the same area. When the area limited by a curve can be expressed in a finite number of algebraic terms, the surface is said to be *quadrable.*

We have found (Art. 122) that the differential of any plane area referred to rectangular co-ordinates is

$$dA = ydx, \qquad (1)$$

where A represents the area included between the curve and the axis of X, and x and y are the general co-ordinates of the curve. The differential of the area included between the curve and the axis of Y is

$$dA = xdy. \qquad (2)$$

If either of these formulas can be integrated, we shall have an expression for the value of A, which is the required area.

Hence, to determine the area limited by a curve and either of the co-ordinate axes, we differentiate its equation, and deduce the value of dx or dy, which we substitute in equation (1) or (2). Having obtained an expression for the differential of the area in terms of one variable and its differential, we integrate it, and obtain a general expression for an indefinite portion of the area. To obtain the area of a definite portion, we must take the integral between the limits denoted by those values of the variables which correspond to the extremities of the limiting curve.

Ex. 1. Find the area of the common parabola.

Differentiating the equation $y^2=4ax$, we have

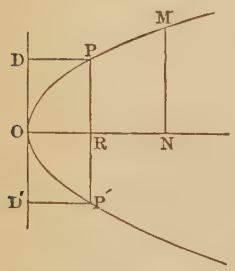
$$dx=\frac{ydy}{2a}.$$

Whence

$$dA=\frac{y^2dy}{2a},$$

and

$$A=\frac{y^3}{6a}+C. \qquad (1)$$



If we estimate the area from the origin O, when $y=0$, $A=0$, and consequently $C=0$, and the corrected integral is

$$A=\frac{y^3}{6a},$$

which equals

$$\frac{y^2\times y}{6a}=\frac{4axy}{6a}=\frac{2xy}{3};$$

that is, the area OPR is $\frac{2}{3}$ODPR; consequently

$$POP'=\tfrac{2}{3}DPP'D',$$

or *the area of the parabola is equal to two thirds of the circumscribed rectangle.*

If it is required to find the area intercepted between two known ordinates, as PR and MN, that is, the area of PMNR, put $PR=b$ and $MN=c$; substitute b and c successively for y in equation (1), and subtract the former result from the latter; that is, integrate between the limits $y=b$ and $y=c$, and we have

$$PMNR=\frac{c^3-b^3}{6a}.$$

Ex. 2. Find the area of a circle.

The equation of the circle is

$$y=(r^2-x^2)^{\frac{1}{2}}.$$

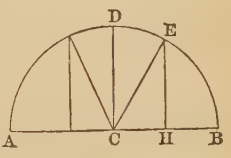
Expanding the second member of this equation by the binomial theorem, we have

$$y=r-\frac{x^2}{2r}-\frac{x^4}{2.4r^3}-\frac{3.x^6}{2.4.6r^5}-\frac{3.5x^8}{2.4.6.8r^7}-\text{ etc.}$$

Multiplying this by dx, and integrating each term separately, we obtain

$$A = rx - \frac{x^3}{2.3r} - \frac{x^5}{2.4.5r^3} - \frac{3x^7}{2.4.6.7r^5} - \frac{3.5x^9}{2.4.6.8.9r^7} - \ldots + C.$$

If we estimate the arc from the point D, when $x=0$, the area is 0, and consequently $C=0$. The preceding series therefore expresses the area of the segment CDEH when x equals CH.



If the arc DE be taken equal to 30°, CH or x becomes $\frac{r}{2}$, and we have

$$CDEH = r^2 \left\{ \frac{1}{2} - \frac{1}{2.3.2^3} - \frac{1}{2.4.5.2^5} - \frac{3}{2.4.6.7.2^7} - \text{etc.} \right\}.$$

The sum of this series may be found as follows:

$$\frac{1}{2.3.2^3} = 0.0208333;$$

$$\frac{1}{2.4.5.2^5} = 0.0007812;$$

$$\frac{3}{2.4.6.7.2^7} = 0.0000698;$$

$$\frac{3.5}{2.4.6.8.9.2^9} = 0.0000085;$$

$$\frac{3.5.7}{2.4.6.8.10.11.2^{11}} = 0.0000012,$$

$$\text{etc.,} \qquad \text{etc.}$$

$$\text{Sum,} \qquad = 0.0216940.$$

Therefore $CDEH = r^2\{0.5 - 0.021694\} = 0.478306r^2$.

But the area of the triangle $CEH = \frac{r^2\sqrt{3}}{8} = 0.216506r^2$. Hence the area of the sector $CDE = 0.261800r^2$, which, multiplied by 12, gives for the area of the whole circle

$$3.1416r^2,$$

which is a little too great, in consequence of the omitted terms of the series, and omitted portions of the decimals.

Ex. 3. Find the area of an ellipse.

The equation to the ellipse referred to its axes is

$$y=\frac{b}{a}\sqrt{a^2-x^2},$$

and consequently the area of the semi-ellipse will be equal to

$$\int y dx=\frac{b}{a}\int(a^2-x^2)^{\frac{1}{2}}dx.$$

But $(a^2-x^2)^{\frac{1}{2}}dx$ is the differential of the area of a circle whose radius is a (Art. 122); hence the area of the ellipse $=\frac{b}{a}\times$ the area of the circumscribing circle. But the area of the circumscribing circle is πa^2; hence the area of the ellipse is equal to

$$\pi a^2\times\frac{b}{a}, \text{ or } \pi ab.$$

Ex. 4. Find the area of an hyperbola.

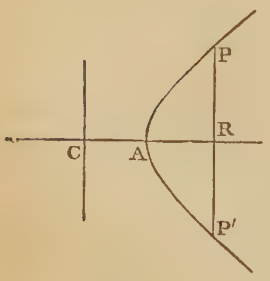
The equation to an hyperbola referred to its centre and axes is

$$y=\frac{b}{a}\sqrt{x^2-a^2};$$

consequently $$dA=ydx=\frac{b}{a}(x^2-a^2)^{\frac{1}{2}}dx.$$

Integrating according to Ex. 5, Art. 197, we obtain

$$A=\frac{bx}{2a}(x^2-a^2)^{\frac{1}{2}}-\frac{ab}{2}\log.\{x+\sqrt{x^2-a^2}\}+C.$$



In order to determine the constant C, make $x=CA=a$, in which case $A=0$, and we have

$$0=-\frac{ab}{2}\log.a+C;$$

that is, $$C=\frac{ab}{2}\log.a.$$

Hence area APR

$$=\frac{xy}{2}-\frac{ab}{2}\log.\frac{x+(x^2-a^2)^{\frac{1}{2}}}{a}.$$

Hence the entire segment APP′ equals

$$xy-ab\log.\left\{\frac{x}{a}+\frac{y}{b}\right\},$$

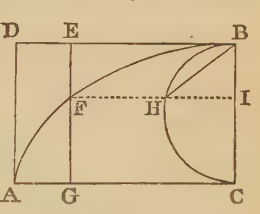
or

$$xy-ab\log.\left\{\frac{ay+bx}{ab}\right\}.$$

Ex. 5. Find the area of an hyperbola whose base is 24, and altitude 10, the transverse axis being 30. *Ans.* 151.6873.

Ex. 6. Find the area of a cycloid.

The area of the figure ABC is most conveniently obtained by first finding the area of the figure ABD, contained between the lines AD, DB, and the convex side of the curve.



Let BC$=2r$, AG$=x$, and FG$=y$, whence FE$=2r-y$. We shall then have

$$d\mathrm{A}=d(\mathrm{ADEF})=(2r-y)dx.$$

But the differential equation of the cycloid (Art. 144) is

$$dx=\frac{ydy}{\sqrt{2ry-y^2}}.$$

Hence

$$d\mathrm{A}=dy\sqrt{2ry-y^2},$$

and

$$\mathrm{A}=\int dy\sqrt{2ry-y^2}+\mathrm{C}.$$

But this is evidently the area of a segment of a circle whose radius is r, and abscissa y (Art. 122); that is, the area of the segment CHI. If we estimate the area of ADEF from AD, and the area of the segment CHI from the point C, they will both be 0 when $y=0$; the constant C will then in each case be 0, and we shall have

$$\mathrm{ADEF}=\mathrm{CHI};$$

and when $y=2r$,

$$\mathrm{ADB}=\text{the semicircle CHB}=\frac{\pi r^2}{2}.$$

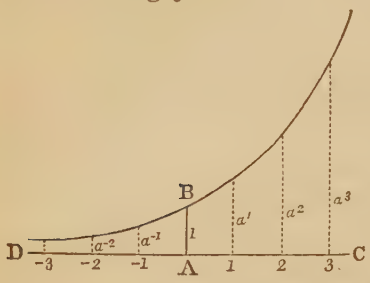
But the area of the rectangle ADBC is equal to

$$\mathrm{AC}\times\mathrm{AD}=\pi r\times 2r=2\pi r^2.$$

Hence the area AFBC$=$ADBC$-$ADB$=\frac{3\pi r^2}{2}=$three times the semicircle CHB; hence *the area included between one*

branch of the cycloid and its base is equal to three times the area of the generating circle.

Ex. 7. Find the area of the logarithmic curve whose equation is $x=\log. y$.



Here $dx=\dfrac{M dy}{y}$;

$$dA=ydx=M dy.$$

Hence $A=My+C$.

If we estimate the area from the line AB, for which $x=0$, and $y=1$, we have

$0=M+C$; hence $C=-M$.

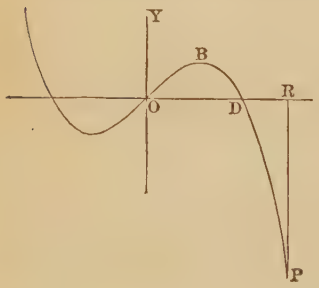
Hence the corrected integral is

$$A=My-M=M(y-1).$$

If $y=0$, we have area $ABD=-M$.

If $y=2$, we have area $=M$.

Ex. 8. Find the area of the curve whose equation is $y=x-x^3$.



Substituting the value of y in the formula Art. 226, we have

$$dA=xdx-x^3dx.$$

Hence $A=\dfrac{x^2}{2}-\dfrac{x^4}{4}+C$.

If we estimate the area from the origin, we shall have $A=0$ when $x=0$, and consequently $C=0$.

If we make $x=OD=1$, we shall have area $OBD=\dfrac{1}{4}$.

To find the area of PDR, we integrate between the limits $x=OD=1$, and $x=OR=\dfrac{3}{2}$, and find

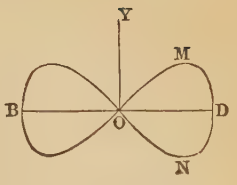
$$\text{area PDR}=\frac{9}{8}-\frac{81}{64}-\frac{1}{4}=-\frac{25}{64}.$$

Ex. 9. Find the area of the curve whose equation is

$$a^2y^2-a^2x^2+x^4=0.$$

$$dA=\frac{x}{a}(a^2-x^2)^{\frac{1}{2}}dx;$$

$$A=-\frac{1}{3a}(a^2-x^2)^{\frac{3}{2}}+C.$$



If we estimate the area from the origin, $A=0$ when $x=0$, and

$$0=-\frac{a^3}{3a}+C, \text{ or } C=\frac{a^2}{3}.$$

Hence the corrected integral is

$$A=\frac{a^2}{3}-\frac{1}{3a}(a^2-x^2)^{\frac{3}{2}}.$$

If we make $x=a$, then

$$A=\text{area OMD}=\frac{a^2}{3}.$$

Hence the oval $OMDN=\frac{2a^2}{3}$, and the area of both ovals $=\frac{4a^2}{3}$.

Ex. 10. Find the area of the curve whose equation is

$$a^4y^2-a^2b^2x^2+b^2x^4=0. \qquad \textit{Ans.}\ \frac{4}{3}ab.$$

227. *Quadrature of polar curves.*

The differential of the area of a polar curve included between a radius vector and the initial line (Art. 126) is

$$dA=\frac{r^2d\theta}{2}.$$

Ex. 1. Find the area of the spiral of Archimedes.

The equation is $r=\frac{b\theta}{2\pi}$ (Anal. Geom., Art. 253).

Hence

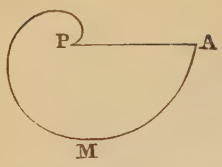
$$d\theta=\frac{2\pi dr}{b};$$

$$dA=\frac{\pi r^2dr}{b};$$

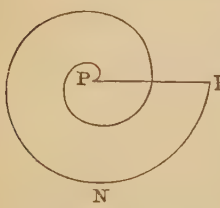
$$A=\frac{\pi r^3}{3b}+C.$$

If we estimate the area from the pole, we have $A=0$ when $r=0$, and consequently $C=0$. The corrected integral is therefore

$$A=\frac{\pi r^3}{3b}.$$



At the end of one revolution of the radius vector, $r = \mathrm{AP} = b$, and A = the area of the first spire = AMP = $\frac{\pi b^2}{3} = \frac{1}{3}$ the area of a circle described with PA as a radius. At the end of the second revolution, $r = 2b$, and the area traced by the radius vector = $\frac{8\pi b^2}{3}$. But in the second revolution the radius vector describes the part PMA a second time; hence to obtain the area PNB, we must subtract that described during the first revolution.



Hence the area PNB = $\frac{7\pi b^2}{3}$.

The area of *the second spire* is $2\pi b^2$, or twice the area of the circle whose radius is equal to the radius vector at the end of the first revolution.

Ex. 2. Find the area of the hyperbolic spiral.

The equation of the hyperbolic spiral is $r\theta = a$.

$$d\mathrm{A} = \frac{a\,dr}{2}.$$

Hence

$$\mathrm{A} = \frac{ar}{2}.$$

The subtangent of this spiral is $-a$ (Ex. 2, Art. 105). Hence the area described by the radius vector from 0 to r is $\frac{ar}{2}$, which is equal to the triangle formed by the radius, the tangent, and the subtangent.

The area included between the curve and two radii vectores r, r' is equal to $\frac{a(r-r')}{2}$.

Ex. 3. Find the area of the logarithmic spiral.

$$\theta = \log.\frac{r}{a};$$

$$d\mathrm{A} = \frac{\mathrm{M}r\,dr}{2};$$

$$A=\frac{Mr^2}{4}+C.$$

If we estimate the area from the pole, where $r=0$, $C=0$, and the corrected integral is $\frac{Mr^2}{4}$.

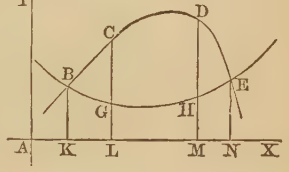
For the Naperian logarithmic spiral, $M=1$, and hence *the area of the Naperian logarithmic spiral is equal to one fourth the square described upon the radius vector.*

The area included between the curve and two radii vectores, b and r, drawn from the pole is $\frac{M}{4}(r^2-b^2)$.

228. *The area included between two curves.* Let BCDE, BGHE be two curves whose equations are respectively

$$y=fx \text{ and } y=Fx.$$

The area LCDM, included by the curve BCE and the two ordinates CL, DM, is given by the formula



$$\int_a^b y\,dx, \text{ or } \int_a^b fx.\,dx,$$

where a and b denote the abscissas of the points C and D.

So also the area LGHM, included by the curve BGE and the same ordinates, is given by the formula

$$\int_a^b Fx.\,dx.$$

Hence the area CDHG is given by the expression

$$A=\int_a^b fx.\,dx-\int_a^b Fx.\,dx, \text{ or } \int_a^b (fx-Fx)dx.$$

If we require the area BCEHB, included between the two curves BCE and BGE, we draw ordinates through the points of intersection B and E, and make a equal to the abscissa AK, and b equal to the abscissa AN.

If the proposed area is bounded by the curves $x=fy$ and

$x=\mathrm{F}y$, and straight lines parallel to the axis of x at distances respectively equal to c and d, we have in a similar manner

$$\mathrm{A}=\int_c^d (fy-\mathrm{F}y)dy.$$

Ex. 1. Find the area included between the straight line $y=2x$ and the parabola $y^2=12x$.

The two lines intersect where $x=0$, $y=0$, and also where $x=3$, $y=6$.

The area of the parabolic space (Art. 226, Ex. 1) is $\frac{2}{3}xy=12$.

The area of the triangle is $\frac{1}{2}xy=9$.

The area of the space included between the proposed lines is therefore 3.

The student should construct a figure illustrating the example.

Ex. 2. Find the area included between the straight line $y=2x-8$ and the parabola $y^2=8x$.

The lines intersect where $x=2$, $y=-4$, and $x=8$, $y=8$.

The area of the required space above the axis of abscissas is $\frac{128}{3}-16=\frac{80}{3}$.

The area of the required space below the axis of abscissas is $\frac{16}{3}+4=\frac{28}{3}$.

The entire area is $\frac{108}{3}=36$.

Ex. 3. Find the area included between the parabola $y^2=8x$ and the circle $y^2=16x-x^2$.

The curves pass through the origin, and meet at the point $x=8$, $y=8$.

If we take only the area which lies on the positive side of the axis of x, we have

$$\mathrm{A}=\int_0^8 \left\{ \sqrt{16x-x^2}-\sqrt{8x} \right\} dx=\frac{64\pi}{4}-\frac{2}{3}\cdot 64=16\pi-\frac{128}{3}.$$

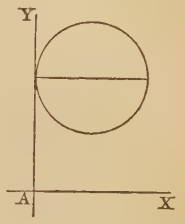
The whole area will therefore be $32\pi-\frac{256}{3}$.

The student should construct the figure.

229. *Area included between different branches of the same curve.* The method of the preceding Article is applicable to finding the area included between two branches of the same curve.

Ex. 1. Find the area included between the two branches of the curve $y^2+x^2-4x-8y+16=0$.

We find $y=4\pm\sqrt{4x-x^2}$.



Hence we may put

$$fx=4+\sqrt{4x-x^2};$$

$$Fx=4-\sqrt{4x-x^2}.$$

Then $fx-Fx=2\sqrt{4x-x^2}$,

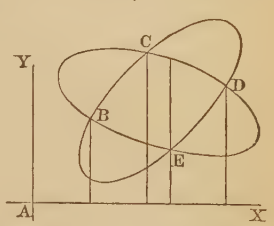
and the complete area of the curve is

$$\int_0^4 2\sqrt{4x-x^2}.dx=4\pi.$$

Ex 2. Find the area included between the opposite branches of an hyperbola, its transverse axis, and any line parallel to it.

$$Ans.\ xy+ab\ \log.\left\{\frac{ay+bx}{ab}\right\}.$$

The preceding method may be extended so as to give the area included between any number of curve lines. If we have two closed curves intersecting each other, as in the figure, and we wish to determine the area of BCDE, we must determine the points of intersection from the equations of the curves, and integrate for each portion of the bounding lines of the desired area. The area may be divided by straight lines parallel to the axes into parts the values of which can be separately obtained by means of the preceding principles.



230. *Area referred to oblique axes.* Hitherto we have supposed the axes to be rectangular; but if they are oblique, and inclined at an angle ω, the formula in Art. 122 becomes

$$dA=\sin.\omega ydx.$$

Hence $A=\sin.\omega\int ydx.$

For example, the equation to the parabola when the axes are a tangent to the curve and a diameter drawn through the point of contact, is $y^2=4a'x$.

Hence the area included between the curve, the axis of x, and an ordinate at the point for which $x=c$, is

$$\sin.\,\omega\int_0^c\sqrt{4a'x}\,.\,dx=\frac{4\,\sin.\,\omega\sqrt{a'}c^{\frac{3}{2}}}{3};$$

that is, two thirds of the parallelogram which has the abscissa c and the ordinate at its extremity for adjacent sides.

Rectification of Plane Curves.

231. *To rectify a curve* is to determine its length, or to obtain a right line equal in length to any assigned portion of the curve. When the length of an arc of a curve can be expressed in a finite number of algebraic terms, the curve is said to be *rectifiable.*

We have found (Art. 121) that the differential of an arc of any plane curve referred to rectangular co-ordinates is

$$d\mathrm{L}=\sqrt{dx^2+dy^2}, \qquad (1)$$

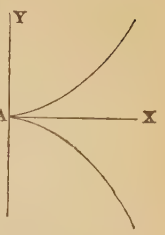
where L represents the length of the curve, and x and y the general co-ordinates. If this formula can be integrated, we shall have an expression for the value of L, which is the length of the arc.

Hence, in order to rectify a curve given by its equation, *we differentiate its equation, and deduce the value of dx or dy, which we substitute in equation* (1). Having obtained an expression for the differential of the arc in terms of one variable and its differential, we integrate it, and obtain a general expression for an indefinite portion of the arc. To obtain the length of a definite portion of the curve, we must take the integral between the limits denoted by those values of the variables which correspond to the extremities of the given portion (Art. 165).

Ex. 1. Rectify the semi-cubical parabola whose equation is $y^2=ax^3$ (Anal. Geom., Art. 237).

Differentiating the given equation, we obtain

$$dy=\frac{3ax^2dx}{2y}.$$



Whence $$dy^2=\frac{9axdx^2}{4}.$$

Substituting this value in equation (1), we have

$$d\mathrm{L}=\left(dx^2+\frac{9axdx^2}{4}\right)^{\frac{1}{2}}=\left(1+\frac{9ax}{4}\right)^{\frac{1}{2}}dx.$$

Integrating by Art. 158, we have

$$\mathrm{L}=\frac{(4+9ax)^{\frac{3}{2}}}{27a}+\mathrm{C}.$$

To determine the constant C, we may reckon the length of the curve from the origin, where L = 0 when $x=0$, and we have

$$0=\frac{8}{27a}+\mathrm{C}.$$

Whence $$\mathrm{C}=-\frac{8}{27a}.$$

Consequently the corrected integral is

$$\mathrm{L}=\frac{(4+9ax)^{\frac{3}{2}}-8}{27a},$$

which expresses the length of an arc of the curve from the origin to the point whose abscissa is x.

Ex. 2. Rectify the curve whose equation is $y^2=\frac{x^3}{4}$, and determine the length of the curve from the origin to the point whose abscissa is 10. *Ans.* 19.0248.

Ex. 3. Rectify the common parabola whose equation is

$$y^2=4ax.$$

Differentiating, we find $dx^2=\frac{y^2dy^2}{4a^2}$.

Whence $$d\mathrm{L}=\frac{dy}{2a}(y^2+4a^2)^{\frac{1}{2}}.$$

Integrating as in Ex. 5, Art. 197, we obtain

$$\mathrm{L}=\frac{y\sqrt{y^2+4a^2}}{4a}+a\log.(y+\sqrt{y^2+4a^2})+\mathrm{C}.$$

If we estimate the arc from the vertex of the parabola, we shall have $y=0$ when $L=0$.

Hence $0=a\log.(2a)+C$, or $C=-a\log.(2a)$,

and the corrected integral is

$$L=\frac{y\sqrt{y^2+4a^2}}{4a}+a\log.\left\{\frac{y+\sqrt{y^2+4a^2}}{2a}\right\}.$$

Ex. 4. Determine the length of the arc of a parabola cut off by a double ordinate to the axis whose length is 12, the abscissa being 2. *Ans.* 12.8374.

NOTE.—Nap. log. = com. log. × 2.3025.

Ex. 5. Rectify the circle.

Differentiating the equation $x^2+y^2=r^2$, we have

$$dy^2=\frac{x^2dx^2}{y^2}.$$

Whence $$dL=\frac{rdx}{\sqrt{r^2-x^2}}.$$

Putting $r=1$, and developing in a series, we have

$$dL=dx+\frac{x^2dx}{2}+\frac{1.3x^4dx}{2.4}+\frac{1.3.5x^6dx}{2.4.6}+\text{ etc.}$$

Integrating, we obtain

$$L=x+\frac{x^3}{2.3}+\frac{1.3x^5}{2.4.5}+\frac{1.3.5x^7}{2.4.6.7}+\ldots+C. \qquad (1)$$

If we measure the arc from the axis of ordinates, we shall have $L=0$ when $x=0$, and therefore $C=0$.

If we make $x=r=1$, we shall have for the length of a quadrant

$$\frac{\pi}{2}=1+\frac{1}{2.3}+\frac{1.3}{2.4.5}+\frac{1.3.5}{2.4.6.7}+\text{ etc.}$$

This series converges so slowly that it requires a very large number of terms to furnish an accurate value of π.

If in equation (1) we make $x=\frac{1}{2}$, then L will correspond to an arc of 30°, that is, to $\frac{\pi}{6}$, and we shall have

$$\frac{\pi}{6}=\frac{1}{2}+\frac{1}{2.3.2^3}+\frac{1.3}{2.4.5.2^5}+\frac{1.3.5}{2.4.6.7.2^7}+\ldots=0.523599.$$

Hence $$\pi=3.14159.$$

232. A better method of obtaining the value of π is to find the length of the arc in terms of its tangent.

We have found (Art. 53) that if L represents an arc of a circle, and x its tangent, we have

$$dL=\frac{dx}{1+x^2}.$$

Developing by the binomial formula, we have

$$dL=dx(1-x^2+x^4-x^6+\text{ etc.}).$$

Integrating and determining C as before, we obtain

$$L=x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\frac{x^9}{9}-\text{ etc.} \qquad (2)$$

If we take L equal to an arc of 30°, its tangent will be equal to $\sqrt{\frac{1}{3}}$, which, being substituted for x in formula (2), gives

$$\frac{\pi}{6}=\sqrt{\frac{1}{3}}\times\left(1-\frac{1}{3.3}+\frac{1}{5.3^2}-\frac{1}{7.3^3}+\frac{1}{9.3^4}-\ldots\right)=0.523599.$$

A still more convenient formula for computing the value of π has already been given in Art. 58.

Ex. 6. Rectify the ellipse.

For this purpose we employ the equation (Anal. Geom., Art. 127) $\qquad y^2=(1-e^2)(a^2-x^2).$

Differentiating, we obtain

$$dy^2=\frac{(1-e^2)x^2}{a^2-x^2}dx^2.$$

Substituting this value in the differential of the arc, we obtain

$$dL=dx\sqrt{1+\frac{(1-e^2)x^2}{a^2-x^2}}=\frac{dx\sqrt{a^2-e^2x^2}}{\sqrt{a^2-x^2}}$$

$$=\frac{adx}{\sqrt{a^2-x^2}}\sqrt{1-\frac{e^2x^2}{a^2}}.$$

Developing $\sqrt{1-\frac{e^2x^2}{a^2}}$ in a series, we obtain

$$dL=\frac{adx}{\sqrt{a^2-x^2}}\left\{1-\frac{e^2x^2}{2a^2}-\frac{e^4x^4}{2.4a^4}-\frac{3e^6x^6}{2.4.6a^6}-\text{ etc.}\right\}.$$

The several terms of this series may be integrated by Formula A, Art. 192, and we obtain

$$L=X_0-\frac{e^2}{2a}X_2-\frac{e^4}{2.4a^3}X_4-\frac{3e^6}{2.4.6a^5}X_6-\ldots+C, \quad (1)$$

where X_0 represents the circular arc whose sine is $\frac{x}{a}$.

$$X_2=\frac{aX_0}{2}-\frac{x}{2}\sqrt{a^2-x^2};$$

$$X_4=\frac{3a^2}{4}X_2-\frac{x^3}{4}\sqrt{a^2-x^2};$$

$$X_6=\frac{5a^2}{6}X_4-\frac{x^5}{6}\sqrt{a^2-x^2}, \text{ etc.}$$

If we estimate the arc of the ellipse from the extremity of the conjugate axis, we shall have $L=0$ when $x=0$, and consequently $C=0$.

In order to obtain one fourth of the circumference of the ellipse, we must integrate between the limits $x=0$ and $x=a$. But when $x=a$, $\sqrt{a^2-x^2}=0$; hence the values of the quantities X_2, X_4, etc., become

$$X_2=\frac{a}{2}X_0;$$

$$X_4=\frac{3a^2}{4}X_2=\frac{3a^3}{2.4}X_0;$$

$$X_6=\frac{5a^2}{6}X_4=\frac{3.5a^5}{2.4.6}X_0, \text{ etc.},$$

and consequently equation (1) becomes

$$L=X_0\left\{1-\frac{e^2}{2.2}-\frac{3e^4}{2.2.4.4}-\frac{3.3.5e^6}{2.2.4.4.6.6}-\text{ etc.}\right\},$$

which expresses the length of one fourth of the circumference of the ellipse, and X_0 represents one fourth of the circumference of the circle whose radius is a. Hence the entire circumference of the ellipse is equal to

$$2\pi a\left\{1-\frac{e^2}{2.2}-\frac{3e^4}{2.2.4.4}-\frac{3.3.5e^6}{2.2.4.4.6.6}-\text{ etc.}\right\}.$$

This series is always converging, and converges very rapidly when the eccentricity is small.

Ex. 7. Determine the circumference of an ellipse whose two axes are 24 and 18 inches. *Ans.* 66.31056.

Ex. 8. Rectify the cycloid.

The differential equation of the cycloid (Art. 144) is

$$dx=\frac{ydy}{\sqrt{2ry-y^2}}.$$

Substituting this value of dx in the differential of the arc, we obtain

$$dL=dy\sqrt{\frac{2ry}{2ry-y^2}}=(2r)^{\frac{1}{2}}(2r-y)^{-\frac{1}{2}}dy.$$

Integrating by Art. 158, we obtain

$$L=-2(2r)^{\frac{1}{2}}(2r-y)^{\frac{1}{2}}+C.$$

If we estimate the arc from the point B, where $y=2r$, we shall have $0=0+C$, or $C=0$,

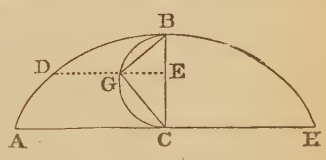
and the corrected integral is

$$L=-2\sqrt{2r(2r-y)},$$

which expresses the length of the arc of the cycloid from B to any point D whose co-ordinates are x and y. But in the right-angled triangle BGC

$BG=\sqrt{BC\times BE}=\sqrt{2r(2r-y)}$.

Hence arc $BD=-2BG$,



or *the arc of a cycloid, estimated from the vertex of the axis, is equal to twice the corresponding chord of the generating circle.* Hence the arc BDA is equal to twice the diameter BC, and the entire arc ADBH is equal to *four times the diameter of the generating circle.*

233. *Rectification of plane curves referred to polar co-ordinates.*

The differential of an arc of a curve referred to polar co-ordinates (Art. 125) is

$$dL=\sqrt{dr^2+r^2d\theta^2}. \qquad (1)$$

In order to rectify a polar curve, we differentiate its equation, and deduce the value of dr or $d\theta$, which we substitute in equation (1). Having obtained a formula with but one variable, we integrate it, and obtain an expression for an indefinite portion of the arc.

Ex. 1. Rectify the logarithmic spiral whose equation is

$$\theta = \log.\frac{r}{a} = \log. r - \log. a.$$

Differentiating, we have

$$d\theta = \frac{M dr}{r}.$$

Hence, by substitution, we find

$$dL = dr\sqrt{1+M^2},$$

and

$$L = r\sqrt{1+M^2} + C.$$

If we estimate the arc from the pole, so that $L=0$ when $r=0$, the constant $C=0$, and the corrected integral is

$$L = r\sqrt{1+M^2}.$$

For the Naperian system of logarithms $M=1$, and hence $L=r\sqrt{2}$; that is, in the Naperian logarithmic spiral *the length of an arc estimated from the pole to any point of the curve is equal to the diagonal of a square described on the radius vector.*

If the radius vector revolves in the negative direction, it generates a portion of the spiral which slowly approaches the pole, but can not reach the pole until after an infinite number of revolutions. In this case the length of an infinite number of spires of the curve is finite (see Anal. Geom., p. 193).

Ex. 2. Rectify the spiral of Archimedes whose equation is

$$r = a\theta.$$

Differentiating, we have $d\theta = \frac{dr}{a}$.

Hence

$$dL = \frac{(r^2+a^2)^{\frac{1}{2}} dr}{a},$$

and

$$L = \frac{1}{a}\int (r^2+a^2)^{\frac{1}{2}} dr.$$

The process of integration is the same as that used in rectifying the common parabola (Ex. 3, Art. 231); hence

$$L=\frac{r\sqrt{r^2+a^2}}{2a}+\frac{a}{2}\log.\left\{\frac{r+\sqrt{r^2+a^2}}{a}\right\},$$

where the arc is estimated from the pole.

QUADRATURE OF SURFACES OF REVOLUTION.

234. We have found (Art. 123) that the differential of the surface of a solid of revolution, when the generating curve revolves about the axis of X, is

$$dS=2\pi y\sqrt{dx^2+dy^2}. \qquad (1)$$

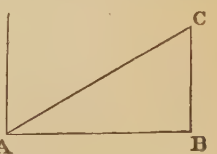
If the curve revolves about the axis of Y, the equation is

$$dS=2\pi x\sqrt{dx^2+dy^2}. \qquad (2)$$

In order to determine the surface of a solid of revolution, we differentiate the equation of the generating curve, and deduce from it the values of y and dy in terms of x and dx, which we substitute in formula (1) or (2). Having obtained an expression for the differential of the surface in terms of one variable and its differential, we integrate it, and obtain a general expression for an indefinite portion of the proposed surface. To obtain the area of a definite portion of the surface, we must take the integral between the limits denoted by those values of the variables which correspond to the boundaries of the proposed surface.

Ex. 1. Determine the convex surface of a cone.

If the right-angled triangle ABC be revolved about AB, the hypothenuse AC will describe the convex surface of a right cone.



If the origin of co-ordinates be at A, the equation of AC is

$$y=mx.$$

Whence $dy=m\,dx,$

and $dS=2\pi mx\,dx\sqrt{1+m^2}.$

Consequently $S=\pi mx^2\sqrt{1+m^2}+C.$

If we estimate the area from the vertex, where $x=0$, we have $C=0$, and the corrected integral is

$$S=\pi m x^2\sqrt{1+m^2}.$$

Making $x=AB=h$, we have the convex surface of the cone whose altitude is h,

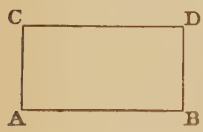
$$S=\pi m h^2\sqrt{1+m^2}.$$

Put $b=BC$, the radius of the base. Then $m=\frac{b}{h}$, and we have

$$\text{convex surface of cone}=\pi b\sqrt{h^2+b^2}=2\pi b\times\frac{AC}{2};$$

that is, *the convex surface of a cone is equal to the circumference of its base into half the slant height.*

Ex. 2. Determine the convex surface of a cylinder.



If the rectangle ABDC be revolved about the side AB, the side CD will describe the convex surface of a right cylinder. Let $CA=b$; then the equation of the line CD will be

$$y=b, \text{ and } dy=0.$$

Hence $$dS=2\pi b dx,$$

and $$S=2\pi bx+C.$$

If we estimate the area from the origin A, we shall have $S=0$ when $x=0$, and consequently $C=0$.

If we make $x=AB=h$, we have the convex surface of the cylinder $=2\pi bh$; that is, *the convex surface of a cylinder is equal to the circumference of its base into its altitude.*

Ex. 3. Determine the surface of a sphere.

The equation of the generating circle, referred to the centre as the origin, is

$$x^2+y^2=r^2.$$

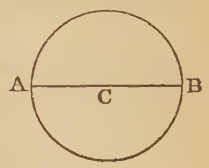
Whence $$dy^2=\frac{x^2dx^2}{y^2}.$$

Hence $$dS=2\pi y\sqrt{dx^2+\frac{x^2dx^2}{y^2}};$$

$$=2\pi r dx;$$

and $$S=2\pi rx+C.$$

If we estimate the surface from a great circle perpendicular to AB, C will equal 0; and if we make $x=r$, we have for the surface of a hemisphere $2\pi r^2$, and for the surface of the sphere $4\pi r^2$; that is, *the surface of a sphere is equal to four of its great circles.*



The surface included between the planes determined by $x=x_1$ and $x=x_2$

is $$2\pi r(x_2-x_1).$$

Hence the area of a zone of a sphere is equal to the product of its altitude by the circumference of a great circle.

Ex. 4. Determine the surface of a paraboloid.

A paraboloid is a solid described by the revolution of an arc AC of a parabola about its axis AB.

The equation of the parabola is $y^2=4ax$.

Whence $$dx^2=\frac{y^2dy^2}{4a^2}.$$

Hence $$dS=\frac{\pi y}{a}(4a^2+y^2)^{\frac{1}{2}}dy,$$

and $$S=\frac{\pi}{3a}(4a^2+y^2)^{\frac{3}{2}}+C.$$

If we estimate the surface from the vertex, we shall have $S=0$ when $y=0$, and consequently

$$0=\frac{8\pi a^2}{3}+C.$$

Whence $$C=-\frac{8\pi a^2}{3};$$

and the corrected integral is

$$\text{surface}=\frac{\pi}{3a}\{(4a^2+y^2)^{\frac{3}{2}}-8a^3\},$$

or $$\frac{8\pi a^{\frac{1}{2}}}{3}\{(a+x)^{\frac{3}{2}}-a^{\frac{3}{2}}\}.$$

Ex. 5. Determine the convex surface of a paraboloid whose axis is 20, and the diameter of whose base is 60.

Ans. 3848.451.

Ex. 6. Determine the surface of a prolate spheroid.

A prolate spheroid is supposed to be generated by the revolution of an ellipse about its transverse axis.

$$dS = 2\pi y\sqrt{dx^2+dy^2} = 2\pi y dL.$$

But in Ex. 6, Art. 232, we have found

$$dL = \frac{adx}{\sqrt{a^2-x^2}}\left\{1-\frac{e^2x^2}{2a^2}-\frac{e^4x^4}{2.4a^4}-\frac{3e^6x^6}{2.4.6a^6}-\text{ etc.}\right\}.$$

Hence

$$dS = \frac{2\pi aydx}{\sqrt{a^2-x^2}}\left\{1-\frac{e^2x^2}{2a^2}-\frac{e^4x^4}{2.4a^4}-\frac{3e^6x^6}{2.4.6a^6}-\text{ etc.}\right\}.$$

But

$$\frac{ay}{\sqrt{a^2-x^2}} = b.$$

Hence

$$dS = 2\pi bdx\left\{1-\frac{e^2x^2}{2a^2}-\frac{e^4x^4}{2.4a^4}-\frac{3e^6x^6}{2.4.6a^6}-\text{ etc.}\right\};$$

and integrating each term separately, we obtain

$$S = 2\pi bx\left\{1-\frac{e^2x^2}{2.3a^2}-\frac{e^4x^4}{2.4.5a^4}-\frac{3e^6x^6}{2.4.6.7a^6}-\cdots\right\}+C.$$

Integrating between the limits $x=0$ and $x=a$, we shall obtain half the surface of the spheroid,

$$=2\pi ab\left\{1-\frac{e^2}{2.3}-\frac{e^4}{2.4.5}-\frac{3e^6}{2.4.6.7}-\frac{3.5e^8}{2.4.6.8.9}-\text{ etc.}\right\}.$$

The surface of a spheroid may be expressed by means of trigonometrical functions without expansion into series; but the formula here given is generally more convenient for computation.

Ex. 7. Determine the surface of a prolate spheroid whose axes are 50 and 40. *Ans.* 5882.638.

Ex. 8. Determine the surface of an oblate spheroid.

An oblate spheroid is supposed to be generated by the revolution of an ellipse about its conjugate axis.

This problem may be solved in a manner similar to Ex. 6.

If we put $\varepsilon^2=\frac{e^2}{1-e^2}$, we shall obtain the entire surface equal to

$$4\pi ab\left\{1+\frac{\varepsilon^2}{2.3}-\frac{\varepsilon^4}{2.4.5}+\frac{3\varepsilon^6}{2.4.6.7}-\frac{3.5\varepsilon^8}{2.4.6.8.9}+\text{ etc.}\right\}.$$

Ex. 9. Determine the surface of the earth, assuming it to be an oblate spheroid whose axes are 7900 and 7926 miles.

Ans. 196,928,025 square miles.

Ex. 10. Determine the surface generated by the revolution of a cycloid about its base.

We have found in Ex. 8, Art. 232,

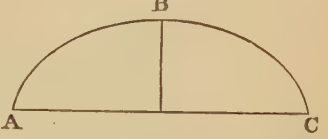
$$dL=(2r)^{\frac{1}{2}}(2r-y)^{-\frac{1}{2}}dy.$$

Hence $dS=2\pi(2r)^{\frac{1}{2}}y(2r-y)^{-\frac{1}{2}}dy=2\pi(2r)^{\frac{1}{2}}y^{\frac{3}{2}}(2ry-y^2)^{-\frac{1}{2}}dy$, the integral of which will give the value of the surface required. But according to Ex. 10, Art. 197,

$$\int y^{\frac{3}{2}}(2ry-y^2)^{-\frac{1}{2}}dy=-\frac{8r}{3}(2r-y)^{\frac{1}{2}}-\frac{2y}{3}(2r-y)^{\frac{1}{2}}.$$

Hence $S=2\pi(2r)^{\frac{1}{2}}\left\{-\frac{8r}{3}(2r-y)^{\frac{1}{2}}-\frac{2y}{3}(2r-y)^{\frac{1}{2}}\right\}+C.$

If we estimate the surface from the plane passing through B perpendicular to AC, we shall have $S=0$ when $y=2r$, and consequently $C=0$.



Taking the area between the limits $y=0$ and $y=2r$, we have

$$\text{half the required surface}=\frac{32\pi r^2}{3}.$$

Hence $$\text{the entire surface}=\frac{64\pi r^2}{3};$$

that is, *the surface generated by a cycloid revolved about its base is equal to* $\frac{64}{3}$ *times the area of the generating circle.*

Ex. 11. Determine the surface generated by the revolution of a cycloid about the tangent at its vertex.

Ans. $\frac{32}{3}\pi r^2$.

Ex. 12. Determine the surface generated by the revolution of a cycloid about its axis.

Ans. $8\pi r^2\left(\pi-\frac{4}{3}\right)$.

CUBATURE OF SOLIDS OF REVOLUTION.

235. The *cubature* of a solid is finding the measure of its solid contents, or finding a cube equivalent to the proposed solid.

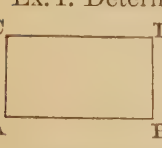
We have found (Art. 124) that the differential of the volume of a solid of revolution is

$$dV=\pi y^2dx, \qquad (1)$$

where x and y are the co-ordinates of the generating curve, and the axis of X is the axis of revolution.

To obtain the cubature of any particular solid, we differentiate the equation of the generating curve, and deduce from it the value of dx in terms of y and dy, or the value of y^2 in terms of x, which we substitute in formula (1). The integral will be an expression for an indefinite portion of the solid.

Ex. 1. Determine the volume of a right cylinder.



The equation of CD will be

$$y=AC=b.$$

Hence $dV=\pi y^2dx=\pi b^2dx,$

and $V=\pi b^2x+C.$

Taking the integral between the limits $x=0$ and $x=AB=h$, we have $V=\pi b^2h$;

that is, *the volume of a right cylinder is equal to the product of its base by its altitude.*

Ex. 2. Determine the volume of a right cone.

As in Ex. 1, Art. 234, $y=mx.$

Hence $dV=\pi y^2dx=\pi m^2x^2dx,$

and $V=\frac{\pi m^2x^3}{3}+C.$

If we estimate the volume from the vertex of the cone, $C=0$.

Let $x=h.$

Then the entire cone $=\frac{\pi m^2h^3}{3}=\frac{\pi b^2h}{3}$;

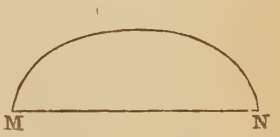
that is, *the volume of a right cone is equal to the area of its*

base into one third of its altitude—that is, *is equal to one third of the circumscribing cylinder.*

Ex. 3. Determine the volume of a prolate spheroid.

The equation of an ellipse is

$$y^2=\frac{b^2}{a^2}(a^2-x^2).$$



Whence $dV=\frac{\pi b^2}{a^2}(a^2-x^2)dx,$

and $$V=\frac{\pi b^2}{a^2}\left(a^2x-\frac{x^3}{3}\right)+C.$$

If we estimate the volume from the plane passing through the centre perpendicular to the transverse axis, we shall have $V=0$ when $x=0$, and consequently $C=0$, and the corrected integral is

$$\frac{\pi b^2}{a^2}\left(a^2x-\frac{x^3}{3}\right).$$

Making $x=a$, we obtain for one half the volume

$$\frac{2\pi b^2 a}{3},$$

and for the entire spheroid

$$\frac{4\pi b^2 a}{3}=\frac{2\pi b^2}{3}\times 2a;$$

that is, *the volume of a prolate spheroid is equal to two thirds of the circumscribing cylinder.*

The volume of a segment of a prolate spheroid included between two planes perpendicular to the transverse axis, one of which passes through the centre, is

$$\frac{\pi b^2 x}{a^2}\left(a^2-\frac{x^2}{3}\right).$$

Ex. 4. Determine the volume of an oblate spheroid.

$$dV=\pi x^2 dy=\frac{\pi a^2}{b^2}(b^2-y^2)dy.$$

Hence $$V=\frac{\pi a^2}{b^2}\left(b^2y-\frac{y^3}{3}\right)+C,$$

which, between the limits $y=-b$ and $y=+b$, gives

$$\frac{4\pi a^2 b}{3}=\frac{2\pi a^2}{3}\times 2b;$$

that is, *two thirds of the circumscribed cylinder.*

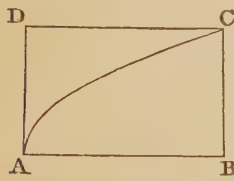
Hence

the prolate spheroid : the oblate spheroid : : $\frac{4\pi b^2 a}{3} : \frac{4\pi a^2 b}{3} : : b : a.$

If in either expression we make $a = b$, we obtain

the volume of a sphere $= \frac{4\pi r^3}{3}$.

Ex. 5. Determine the volume of a paraboloid.



$$dV = 4\pi a x dx.$$

Hence $V = 2\pi a x^2 + C.$

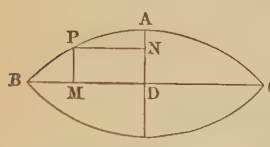
If we estimate the volume from A, $C = 0$, and we have

the volume $= 2\pi a x^2 = \frac{1}{2}\pi y^2 x$;

that is, half the area of the base × perpendicular height; or *half the volume of the circumscribed cylinder.*

Ex. 6. Determine the volume of a paraboloid whose axis is 30, and the diameter of its base 40. *Ans.* 18849.556.

Ex. 7. Determine the volume of a parabolic spindle generated by the revolution of a parabola about its double ordinate BC.



Let AD $= h$, BD $= b$, AN $= x$, and PN $= y$. Then PM $= h - x$, and the area of the circle described by the point P in its revolution $= \pi(h-x)^2$.

Hence $dV = \pi(h-x)^2 dy = \pi h^2 dy - 2\pi h x dy + \pi x^2 dy.$

Substituting for x its value $\frac{y^2}{4a}$, we have

$$dV = \pi h^2 dy - \frac{\pi h y^2 dy}{2a} + \frac{\pi y^4 dy}{16a^2}.$$

Hence $$V = \pi h^2 y - \frac{\pi h y^3}{6a} + \frac{\pi y^5}{80a^2} + C,$$

or $$V = \pi h^2 y - \frac{2}{3}\pi h x y + \frac{\pi}{5} y x^2 + C.$$

Taking the integral between the limits $x = 0$ and $x = h$, we have

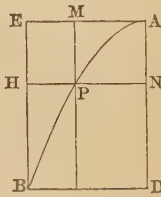
$$V = \pi h^2 b - \frac{2}{3}\pi h^2 b + \frac{\pi}{5} h^2 b = \frac{8\pi h^2 b}{15};$$

that is, *the volume of the spindle is eight fifteenths of the circumscribed cylinder.*

Ex. 8. Determine the volume of a parabolic spindle whose length is 80, and whose greatest diameter is 32.

Ans. 34314.569.

Ex. 9. Determine the volume of the solid generated by an arc of a parabola revolving about the tangent at its vertex.



The area of the circle described by the point P in its revolution about AE is πx^2. Hence

$$dV = \pi x^2 dy = \frac{\pi y^4 dy}{16a^2}.$$

Hence $$V = \frac{\pi y^5}{80a^2} = \frac{\pi x^2 y}{5} + C.$$

Taking the integral between the limits $x=0$ and $x=h$, we have

$$V = \frac{\pi h^2 b}{5}, \text{ or } \textit{one fifth of the circumscribed cylinder.}$$

Ex. 10. Determine the volume of the solid generated by an arc of a parabola revolving about EB parallel to the axis AD.

The area of the circle described by the point P in its revolution about EB is $\pi(b-y)^2$. Hence

$$\begin{aligned} dV &= \pi(b-y)^2 dx \\ &= \pi b^2 dx - 2\pi b y dx + \pi y^2 dx \\ &= \pi b^2 dx - 4\pi b a^{\frac{1}{2}} x^{\frac{1}{2}} dx + 4\pi a x dx. \end{aligned}$$

Hence $$\begin{aligned} V &= \pi b^2 x - \frac{8}{3}\pi b a^{\frac{1}{2}} x^{\frac{3}{2}} + 2\pi a x^2 + C \\ &= \pi b^2 x - \frac{4}{3}\pi b x y + \frac{\pi}{2} x y^2 + C. \end{aligned}$$

Taking the integral between the limits $x=0$ and $x=h$, we have

$$V = \pi b^2 h - \frac{4}{3}\pi b^2 h + \frac{\pi}{2} b^2 h,$$

or $V = \frac{\pi b^2 h}{6}$, or *one sixth of the cylinder having the same base and altitude.*

Ex. 11. Determine the volume of the solid generated by the revolution of a cycloid around its base.

The differential equation of the cycloid (Art. 144) is

$$dx = \frac{ydy}{\sqrt{2ry - y^2}}.$$

Hence $$dV = \pi y^2 dx = \frac{\pi y^3 dy}{\sqrt{2ry - y^2}}.$$

This expression may be integrated by Formula E, Art. 196, and we obtain

$$V = -\frac{\pi y^2}{3}(2ry - y^2)^{\frac{1}{2}} + \frac{5\pi r}{3}\int y^2(2ry - y^2)^{-\frac{1}{2}}dy.$$

The integral of $y^2(2ry - y^2)^{-\frac{1}{2}}dy$ has been given in Ex. 9, Art. 197.

The integral must be taken between the limits $y = 0$ and $y = 2r$. When $y = 0$, all the terms of the above expression become 0. When $y = 2r$, the arc of which y is the versed sine becomes πr, and the expression for V reduces to

$$\pi r \times \frac{3r}{2} \times \frac{5\pi r}{3},$$

which equals $$\frac{5\pi^2 r^3}{2};$$

that is, *the volume of the solid is five eighths of the circumscribing cylinder.*

CHAPTER VII.

APPLICATIONS OF THE DIFFERENTIAL AND INTEGRAL CALCULUS TO MECHANICS.

Centre of Gravity.

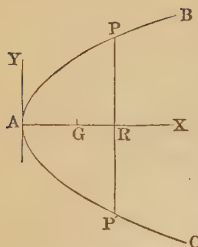
236. *Definitions.* In determining the centre of gravity of a body, it is convenient to regard the body as composed of an indefinitely large number of particles, each of which is indefinitely small, and the weight of the body is the resultant of the weights of the different elementary portions of the body acting in parallel and vertical lines. This resultant passes through a point which is called *the centre of gravity* of the body.

It is proved in treatises on Mechanics that if each particle of a body be multiplied by its distance from any assumed plane, and the sum of the products be divided by the sum of the particles, the quotient will denote the distance of the common centre of gravity from the assumed plane. The product of any particle by its distance from a given plane is called its *moment* with respect to that plane. The *moment of a body* is the sum of the moments of its particles.

237. In all of the following examples bodies will be supposed to be of uniform density. Surfaces, lines, and points are supposed to be material. A material surface is a body whose length and breadth are finite, but whose thickness is indefinitely small; a material line is a body whose length is finite, but its breadth and thickness are indefinitely small; and a material point is a body whose length, breadth, and thickness are indefinitely small.

238. *To find the centre of gravity of a line.*

Let BAC be any plane curve symmetrical with respect to



the axis AX, and let AY be drawn perpendicular to AX. Draw PRP′ perpendicular to AX. Then, since the curve is symmetrical with respect to AX, PR is equal to P′R, and the centre of gravity of the two particles P and P′ will be on the line AX. The same will be true of any two corresponding particles on the two branches of the curve. Hence the centre of gravity of the line BAC must be situated in the line AX. Designate the position of the centre of gravity by G; it is required to find the distance of G from AY.

Let x represent the abscissa, and y the ordinate, for any particle of the line BAC. It is convenient to regard the differential of the line as representing one of the elementary portions or particles of the line. If, then, s represents the line BAC, ds will represent one of the elementary particles of the line. If we multiply this differential by the distance of the particle from AY, the product xds will represent the moment of that particle, and its integral $\int xds$ will represent the sum of the products arising from multiplying each particle by its distance from AY. Dividing this sum by the sum of the particles, or s, we have $\frac{\int xds}{s}$, which represents AG, the distance of the centre of gravity from AY.

If we consider simply the portion of the line comprehended between A and B, the distance of its centre of gravity from AY will be the same as that of the whole line BAC, but its distance from AX will be represented by $\frac{\int yds}{s}$.

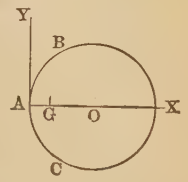
Ex. 1. Determine the centre of gravity of a straight line.

Conceive the line to be placed on AX, with one extremity at the origin A. The moment of any particle of the line is xdx, and the moment of the whole line is $\int xdx$, while the length of the line is x. Hence

$$\text{AG} = \frac{\int xdx}{x} = \frac{\frac{1}{2}x^2 + \text{C}}{x} = \frac{x}{2};$$

that is, the centre of gravity of the line is at its middle point.

In the above expression the constant C is 0, because when $x=0$ the function $=0$. The same remark applies to most of the subsequent examples.



Ex. 2. Determine the centre of gravity of a circular arc $BAC=2s$.

The equation of the circle referred to the axes AX, AY, is

$$y^2=2rx-x^2.$$

Hence
$$dy=\frac{(r-x)dx}{\sqrt{2rx-x^2}},$$

and
$$ds=\sqrt{dx^2+dy^2}=\frac{rdx}{\sqrt{2rx-x^2}}.$$

Hence
$$AG=\frac{\int xds}{s}=\frac{r}{s}\int\frac{xdx}{\sqrt{2rx-x^2}}$$

$$=\frac{r}{s}\{-\sqrt{2rx-x^2}+s\}=r-\frac{ry}{s}.$$

Hence
$$OG=r-AG=\frac{ry}{s};$$

that is, the distance of the centre of gravity from the centre of the circle is a fourth proportional to the arc, the radius, and the chord of the arc.

When the arc is a semi-circumference, the chord $=2r$, and the arc $=\pi r$. Therefore

$$OG=\frac{2r}{\pi}.$$

Ex. 3. Determine the centre of gravity of the arc of a cycloid.

We will assume as the axes of reference the axis of the cycloid and a tangent to the curve at its vertex.

The differential equation of the cycloid is

$$dy=\frac{\sqrt{2ry-y^2}}{y}dx=\sqrt{\frac{2r-y}{y}}dx.$$

But since the origin is at the vertex, substituting $2r-y$ for y, we have

$$dx^2=\frac{2r-y}{y}dy^2.$$

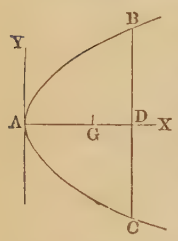
Therefore $$ds=\sqrt{dx^2+dy^2}=dy\sqrt{\frac{2r}{y}}.$$

Also, $$s=2\sqrt{2ry}\ \text{(Ex. 8, Art. 232)}.$$

Hence $$\text{AG}=\frac{\int yds}{s}=\frac{\sqrt{2r}}{2\sqrt{2ry}}\int y^{\frac{1}{2}}dy=\frac{y}{3}.$$

For the entire curve $y=2r$, so that the distance of the centre of gravity of the entire curve from the vertex is equal to *one third of the diameter of the generating circle.*

239. *To find the centre of gravity of a plane surface.*



Let BAC be a plane curve symmetrical with respect to AX; it is required to find the centre of gravity of the surface bounded by this curve and the line BC drawn perpendicular to AX. The distance of the centre of gravity from AY must be the same for the whole figure BAC as for the half figure BAD.

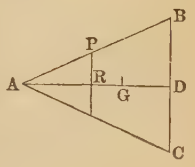
The differential of the area of the segment is ydx. Regarding the surface as a material body, and multiplying the differential of the area by x, the distance of any particle from AY, the product $xydx$ will be the differential of its moment. Hence the moment of the entire surface BAD will be represented by $\int xydx$, and the distance of the centre of gravity from AY will be represented by

$$\frac{\int xydx}{\text{area}}.$$

If we consider simply the area BAD, the distance of its centre of gravity from AY will be the same as that of BAC; but to find the distance of the centre of gravity of BAD from AX, we must multiply ydx by the distance of the centre of gravity of the element of surface from AX, viz. $\frac{1}{2}y$. Hence the differential of the moment of BAD with reference to AX

is $\frac{1}{2}y^2dx$, and the distance of the centre of gravity of BAD from AX will be represented by $\frac{\frac{1}{2}\int y^2dx}{\text{area}}$.

Ex. 1. Determine the centre of gravity of an isosceles triangle.



It is evident that the centre of gravity will be somewhere in the line AD perpendicular to the base.

Let $\quad AD=a,\ BD=b,\ AR=x,\ PR=y.$

Then $$y=\frac{bx}{a}.$$

Hence $$AG=\frac{\int xydx}{\text{area}}=\frac{\int\frac{bx^2}{a}dx}{\frac{1}{2}xy}=\frac{2x}{3};$$

that is, the distance of the centre of gravity from the vertex of the triangle is equal to *two thirds the altitude of the triangle.*

Ex. 2. Determine the centre of gravity of a parabolic area.

Here $\quad y^2=4ax$, and $\text{area}=\frac{2xy}{3}=\frac{4}{3}a^{\frac{1}{2}}x^{\frac{3}{2}};$

$$AG=\frac{\int xydx}{\text{area}}=\frac{\int 2a^{\frac{1}{2}}x^{\frac{3}{2}}dx}{\text{area}}=\frac{\frac{4}{5}a^{\frac{1}{2}}x^{\frac{5}{2}}}{\frac{4}{3}a^{\frac{1}{2}}x^{\frac{3}{2}}}=\frac{3x}{5};$$

that is, the distance of the centre of gravity from the vertex equals three fifths of the axis of the parabola.

Ex. 3. Determine the centre of gravity of a segment of a circle.

We will determine the distance of the centre of gravity from the centre of the circle, instead of from A.

The equation is $\quad y^2=r^2-x^2.$

Hence $$OG=\frac{\int x(r^2-x^2)^{\frac{1}{2}}dx}{\text{area}}=\frac{\frac{1}{3}(r^2-x^2)^{\frac{3}{2}}+C}{\text{area}}.$$

If the segment is a semicircle, then, integrating between $x=0$ and $x=-r$, and placing area $=\frac{1}{4}\pi r^2$, we have

$$OG=\frac{4r}{3\pi}.$$

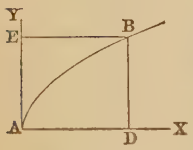
Ex. 4. Determine the centre of gravity of a segment of an ellipse cut off by an ordinate to the transverse axis.

The equation is $y^2=\frac{b^2}{a^2}(a^2-x^2)$.

Hence $$OG=\frac{\int\frac{b}{a}(a^2-x^2)^{\frac{1}{2}}xdx}{\text{area}}=\frac{\frac{b}{3a}(a^2-x^2)^{\frac{3}{2}}+C}{\text{area}}.$$

If the segment is a semi-ellipse, then, integrating between $x=0$ and $x=-a$, and placing area $=\frac{1}{4}\pi ab$, we have

$$OG=\frac{4a}{3\pi}.$$



Ex. 5. ABD is a segment of a parabola cut off by an ordinate to the axis. Determine the distance of its centre of gravity from AX.

Ans. $\frac{3y}{8}$.

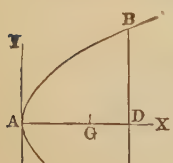
Ex. 6. Determine the distance of the centre of gravity of ABE from AY.

Ans. $\frac{3x}{10}$.

Ex. 7. Determine the distance of the centre of gravity of ABE from AX.

Ans. $\frac{3y}{4}$.

240. *To find the centre of gravity of a surface of revolution.*



The differential of the surface generated by the revolution of AB around AX is $2\pi yds$, and the differential of its moment in reference to the plane passing through AY perpendicular to AX is $2\pi xyds$. Hence the moment of the entire surface generated will be represented by $\int 2\pi xyds$, and the distance of the

centre of gravity from AY will be represented by

$$\frac{\int 2\pi xy\,ds}{\text{surface}}.$$

Ex. 1. Determine the centre of gravity of the convex surface of a right cylinder.

Here $y=b$, and the convex surface of the cylinder $=2\pi bx$.

Hence $$\text{AG}=\frac{\int 2\pi bx\,dx}{2\pi bx}=\frac{x}{2};$$

that is, the centre of gravity is at the middle point of the axis.

Ex. 2. Determine the centre of gravity of the convex surface of a right cone.

Here $y=ax$, and the convex surface of the cone $=\pi y(x^2+y^2)^{\frac{1}{2}}$.

Also, $$ds=(dx^2+dy^2)^{\frac{1}{2}}=dx(a^2+1)^{\frac{1}{2}}.$$

Hence $$\text{AG}=\frac{\int 2\pi xy(a^2+1)^{\frac{1}{2}}dx}{\pi ax^2(a^2+1)^{\frac{1}{2}}}=\frac{2x}{3};$$

that is, the centre of gravity is at a distance from the vertex equal to two thirds the length of the axis.

Ex. 3. Determine the centre of gravity of the surface of a spherical segment.

Here $$y^2=r^2-x^2.$$

Also, $$ds=(dx^2+dy^2)^{\frac{1}{2}}=\left(\frac{r^2}{r^2-x^2}\right)^{\frac{1}{2}}dx=\frac{r\,dx}{y},$$

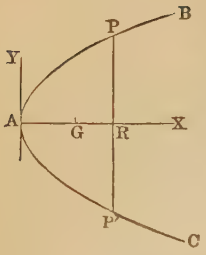
and $$\text{surface}=2\pi rx.$$

Hence $$\text{AG}=\frac{\int 2\pi rx\,dx}{2\pi rx}=\frac{x}{2}.$$

For a hemisphere, $\text{AG}=\frac{r}{2}$; that is, the centre of gravity of the convex surface of a hemisphere is at a distance from the centre equal to half the radius.

241. *To find the centre of gravity of a solid of revolution.*

The differential of the solid generated by the revolution of



AB around AX is $\pi y^2 dx$, and the differential of its moment in reference to a plane passing through AY perpendicular to the plane BAC is $\pi x y^2 dx$. Hence the moment of the solid generated will be represented by $\int \pi x y^2 dx$, and the distance of the centre of gravity from AY will be represented by

$$\frac{\int \pi x y^2 dx}{\text{volume}}.$$

Ex. 1. Determine the centre of gravity of a right cone.

Here $y = ax$, and volume $= \frac{\pi y^2 x}{3}$.

Hence $$\text{AG} = \frac{\int \pi a^2 x^3 dx}{\frac{1}{3}\pi a^2 x^3} = \frac{3x}{4};$$

that is, the distance of the centre of gravity from the vertex is equal to three fourths of the axis.

Ex. 2. Determine the centre of gravity of a paraboloid.

Here $y^2 = 4ax$, and volume $= \frac{1}{2}\pi y^2 x$.

Hence $$\text{AG} = \frac{\int 4\pi a x^2 dx}{2\pi a x^2} = \frac{2x}{3};$$

that is, the distance of the centre of gravity from the vertex is equal to two thirds of the axis.

Ex. 3. Determine the centre of gravity of a segment of a sphere.

Here $y^2 = 2rx - x^2$, and volume $= \pi r x^2 - \frac{1}{3}\pi x^3$.

Hence $$\text{AG} = \frac{\int \pi x(2rx - x^2) dx}{\pi r x^2 - \frac{1}{3}\pi x^3} = \frac{x(8r - 3x)}{4(3r - x)}.$$

When $x = r$, $$\text{AG} = \frac{5r}{8};$$

that is, the centre of gravity of a hemisphere is at a distance from the centre equal to three eighths of the radius.

Ex. 4. Determine the centre of gravity of a segment of a prolate spheroid.

Here $$y^2 = \frac{b^2}{a^2}(2ax - x^2).$$

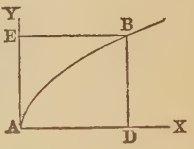
Hence $$AG=\frac{\int\frac{\pi b^2}{a^2}x(2ax-x^2)dx}{\text{volume}}=\frac{\frac{\pi b^2}{a^2}\{\frac{2}{3}ax^3-\frac{1}{4}x^4\}}{\text{volume}}.$$

When $x=a$, $$\text{volume}=\frac{2\pi b^2 a}{3},$$

and $$AG=\frac{5a}{8};$$

that is, the centre of gravity of a hemispheroid is the same as that of the hemisphere whose diameter is the major axis of the spheroid.

Ex. 5. ABD is a segment of a parabola cut off by an ordinate to the axis. If this segment revolve about BD, determine the distance of the centre of gravity of the solid from AX.



Ans. $\frac{5y}{16}$.

Ex. 6. If the segment ABE revolve about AY, determine the distance of the centre of gravity of the solid from AX.

Ans. $\frac{5y}{6}$.

Ex. 7. If the segment ABE revolve about BE, determine the distance of the centre of gravity of the solid from AY.

Ans. $\frac{x}{5}$.

242. *General theorem.* The examples in the six preceding articles will be recognized as special cases of a general theorem, which (using three co-ordinate axes) may be thus stated: *The distance of the centre of gravity of a mass* (m) *from the plane yz* is $$\frac{\int xdm}{\int dm},$$

where dm is the differential of mass, and the two integrals have the same limits depending upon the extent of m. The distances of the same point from the other two co-ordinate planes are $$\frac{\int ydm}{\int dm} \text{ and } \frac{\int zdm}{\int dm}.$$

CENTRE OF OSCILLATION OF A PENDULUM.

243. *A simple pendulum.* A single material particle suspended by a slender wire supposed to have no weight, and oscillating freely about a fixed horizontal axis connected with the other extremity of the wire, is called a simple pendulum.

A compound pendulum. Any material body free to oscillate about a fixed horizontal axis is called a compound pendulum.

A simple pendulum, as above defined, can have no physical existence; but it is convenient to regard a compound pendulum as a collection of simple pendulums.

The axis of a compound pendulum is the straight line drawn through its centre of gravity perpendicular to the horizontal axis about which the pendulum vibrates.

244. *The centre of oscillation* of a pendulum is that point of its axis, or of its axis produced, at which, if the entire mass of the pendulum were collected, its time of vibration would be unchanged.

The length of a compound pendulum is that part of its axis which is included between the axis of suspension and the centre of oscillation, or it is the length of the simple pendulum which vibrates in the same time.

245. *Principle of moments.* The resistance to the motion of a solid body about any axis is called the *moment of inertia* of the body with respect to that axis, and is expressed by the sum of the products of each of the particles of the body into the square of their respective distances from the axis of rotation. Thus, let S be the point of suspension of a pendulum; let A, B, C, etc., denote the component particles of the body, and a, b, c, etc., their respective distances from the axis of suspension. Then

$$Aa^2+Bb^2+Cc^2+ \text{ etc.}$$

is called the *moment of inertia* of the body with respect to the point S.

246. The *moment of the mass* with respect to the axis of suspension is the product arising from multiplying the sum of the particles by the distance of the common centre of gravity from the axis of suspension. If G denote the centre of gravity of the pendulum, and SG the perpendicular from G upon the axis, then $(A+B+C+\ldots)\times SG$ denotes the moment of the mass with respect to the axis.

247. *General formula.* It is proved in treatises on Mechanics that the distance from the centre of oscillation to the axis of suspension is found by dividing the moment of inertia with respect to that axis by the moment of the mass with respect to the same axis. If, then, O represent the centre of oscillation of a pendulum SGD, and SO the distance of O from the axis of suspension, we shall have

$$SO=\frac{Aa^2+Bb^2+Cc^2+\text{ etc.}}{(A+B+C+\ldots)\times SG}.$$

248. If we represent one of the elementary portions of the body, that is, the differential of the body, by dm, and its distance from the axis of suspension by x, then x^2dm will represent its moment of inertia, and $\int x^2dm$ will denote the moment of inertia of all the particles of the body. If we represent the mass of the body by m, and the distance of the centre of gravity from S by g, we shall have

$$SO=\frac{\int x^2dm}{\int xdm}=\frac{\int x^2dm}{mg}.$$

Ex. 1. Find the centre of oscillation of a slender rod or straight line suspended from one of its extremities.

Here, as we regard only the length of the rod, dm will be represented by dx. Hence

$$SO=\frac{\int x^2dx}{mg}=\frac{x^3}{3mg}.$$

If x be taken equal to the whole length of the rod, then

$$m=x, \text{ and } g=\frac{x}{2}.$$

Hence $$SO=\frac{2x}{3};$$

that is, the distance of the centre of oscillation from the axis of suspension is two thirds the length of the rod.

Ex. 2. If a cylindrical rod 58.65 inches long, when suspended from one extremity, make a vibration in one second, what is the length of the seconds' pendulum? *Ans.* 39.1 inches.

Ex. 3. Find the centre of oscillation of a slender rod suspended from any point above its centre of gravity.

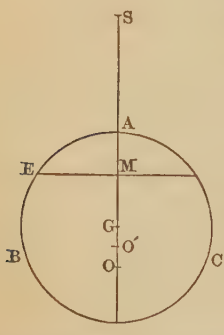
Let S be the point of suspension of the rod AP. The moment of inertia of the particles in SA is $\int x^2dx=\frac{1}{3}SA^3$; and the moment of inertia of the particles in SP is $\frac{1}{3}SP^3$, the sum of which is $\frac{1}{3}(SP^3+SA^3)$. The vibrating body is SP+SA, and the distance of the centre of gravity from $S=\frac{1}{2}(SP-SA)$. Therefore

$$mg=\tfrac{1}{2}(SP^2-SA^2).$$

Hence $$SO=\frac{\frac{1}{3}(SP^3+SA^3)}{\frac{1}{2}(SP^2-SA^2)}=\frac{2}{3}\times\frac{SP^3+SA^3}{SP^2-SA^2}.$$

Ex. 4. If a cylindrical rod 64 inches long, whose axis of suspension is distant 6 inches from one extremity of the rod, make a vibration in one second, what is the length of the seconds' pendulum? *Ans.* 39.13 inches.

249. It is proved in treatises on Mechanics that along a given line through the centre of gravity the distance of the centre of oscillation of a pendulum from the centre of gravity varies inversely as the distance of the point of suspension from the same. If ABC be a solid of revolution, G its centre of gravity, O the centre of oscillation when the body is suspended from A, and O′ the centre of oscillation when it is suspended from S, then

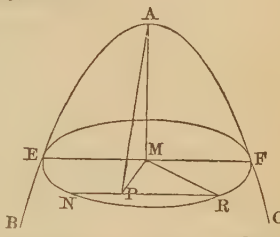


$$GO:GO'::SG:AG.$$

Hence, if we can find the centre of oscillation of the body

when suspended from any point A, we can find the centre of oscillation when it is suspended from any other point situated on the line AG.

250. *Centre of oscillation of a solid of revolution.* Suppose a solid to be generated by the revolution of the curve BAC about its axis AM; it is required to find its centre of oscillation when suspended from its vertex A.



Let ENRF be a circular section perpendicular to the axis AM; let EF be that diameter of the section which is parallel to the axis of suspension; draw NR parallel, and MP perpendicular to EF. Join AP, MR. We wish to find the moment of inertia of the circle ENRF regarded as a plate of matter whose thickness is indefinitely small.

Let $$AM=a,\ MR=r,\ MP=v.$$

Then $$AP^2=a^2+v^2,\ NR=2\sqrt{r^2-v^2},$$

and the differential of the area ENPRF is

$$dA=2dv\sqrt{r^2-v^2}.$$

Hence the moment of inertia of an elementary portion of the circle is

$$2dv\sqrt{r^2-v^2}\times(a^2+v^2),$$

or

$$2a^2dv\sqrt{r^2-v^2}+2v^2dv\sqrt{r^2-v^2}.$$

Hence the moment of inertia of the circle ENRF is

$$a^2A+\int 2v^2dv\sqrt{r^2-v^2},$$

or

$$a^2A+\frac{r^2A}{4}-\frac{v}{2}(r^2-v^2)^{\frac{3}{2}}.$$

When $v=r$, this expression becomes

$$\left(a^2+\frac{r^2}{4}\right)\times\text{semicircle ENRF},$$

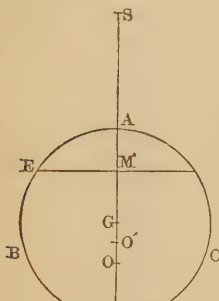
and for the whole circle the expression becomes

$$\left(a^2+\frac{r^2}{4}\right)\pi r^2=\pi a^2r^2+\tfrac{1}{4}\pi r^4.$$

Let $AM=x$, EM or $MR=y$. Then, for the circular section

ENRF the moment of inertia is $\pi x^2y^2+\frac{1}{4}\pi y^4$, and therefore the moment of inertia of the entire solid with respect to its vertex A will be $\int(\pi y^2x^2dx+\frac{1}{4}\pi y^4dx)$, (1)
where the integration extends to the whole solid.

In order to apply this formula to a particular case, we must substitute for y its value in terms of x, as determined from the equation of the curve; the integral will give the moment of inertia of the solid.



Ex. 1. Find the centre of oscillation of a sphere suspended from its vertex A.

Let r = the radius of the sphere, and let O be the centre of oscillation when it is suspended from A.

The equation of the curve is

$$y^2=2rx-x^2.$$

Hence the moment of inertia equals

$$\int\pi x^2dx(2rx-x^2)+\int\tfrac{1}{4}\pi dx(2rx-x^2)^2,$$

which equals $\frac{\pi r^2x^3}{3}+\frac{\pi rx^4}{4}-\frac{3\pi x^5}{20}$.

Let $x=2r$. Then, since $m=\frac{4}{3}\pi r^3$, and $g=r$, by Art. 248 we have

$$\mathrm{AO}=\frac{\frac{8}{3}\pi r^5+4\pi r^5-\frac{24}{5}\pi r^5}{\frac{4}{3}\pi r^4}=\frac{7}{5}r.$$

Hence $\mathrm{GO}=\frac{2}{5}r$.

Ex. 2. Find the centre of oscillation of a sphere suspended from any point S.

Let O′ be the centre of oscillation when suspended from S.

Then, by Art. 249, $\frac{2r}{5}:\mathrm{GO}'::\mathrm{SG}:r.$

Hence $$\mathrm{GO}'=\frac{2}{5}\times\frac{r^2}{\mathrm{SG}}.$$

Ex. 3. A sphere two inches in diameter, suspended by a slender wire whose length, measured from the point of suspension to the centre of the sphere, was 39.1 inches, made one vibration per second. What is the length of the simple seconds' pendulum? *Ans.* 39.1102 inches.

ON THE ATTRACTION OF BODIES.

251. *Units of measurement.* In order to compare forces with each other, it is necessary to employ some standard units of time, space, velocity, etc. *One second* is usually taken for the unit of time, and *one foot* for the unit of space. In treating of the heavenly bodies, it is sometimes, however, more convenient to employ other units of time and space.

Uniform velocity is measured by the number of units of space described in a unit of time; that is, by the number of feet described in one second. When the velocity is variable, it is measured by the *limit* of the ratio of the space described to the time of describing it.

The English *unit of mass* is the standard pound (avoirdupois). The mass of any given substance is denoted by the number of pounds it contains.

252. *Momentum.* The momentum of a moving body, or its *quantity of motion*, is the product obtained by multiplying the mass moved by the velocity with which it is moved, and is expressed numerically by the product of the number of units of mass which it contains, and the number of units of velocity in its motion. Thus, if a body contain five pounds of matter, and move at the rate of eight feet per second, its momentum is represented by the number forty.

253. *Measure of force.* Our simplest conceptions of force are derived from weight. The force with which the earth attracts the unit of mass at a given place is the usual unit of force. But in Mechanics other units of force are often more convenient. A constant force is measured by the momentum it communicates to a body in a unit of time. A variable force is measured at any instant by the momentum it would communicate if it continued constant for a unit of time from the given instant. The *unit of force* is that force which, if continued constant, would impart in a second of time a velocity of one foot per second to a pound of matter.

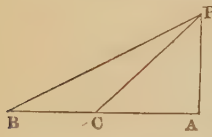
If the mass moved contain m pounds of matter, and if the velocity generated in a unit of time is represented by f, the units of force will be represented by mf.

254. *Law of gravitation.* According to the Newtonian law of gravitation, two particles of matter placed at any sensible distance apart, attract each other with a force which varies directly as the product of their masses, and inversely as the square of their distance from each other.

Suppose, then, a particle to be attracted by all the particles of a body; if we resolve the attraction of each particle into components parallel to fixed rectangular axes, and take the sum of the components which act in a given direction, we shall obtain the resolved attraction of the whole body on the particle in that direction, and can thus ascertain the resultant attraction of the body in magnitude and direction. We shall apply this principle to a few simple cases.

255. *To find the resolved attraction of a uniform straight line which is exerted in a direction perpendicular to the line.*

By a straight line we understand a material prism or cylinder whose section perpendicular to its axis is indefinitely small.



Let AB be the given straight line, and P a particle of matter attracted by AB. Draw PA perpendicular to AB; let $PA=a$; represent AC, a variable part of AB, by x. Then

$$PC=\sqrt{a^2+x^2}.$$

Let m denote the mass of a portion of the given line one unit in length; then mdx will denote the mass of an elementary portion of the line AB. Let m' denote the mass of P; then, by Art. 254, the attraction of the elementary portion at C upon the particle P will be

$$\frac{kmm'dx}{PC^2},$$

where k denotes a constant quantity, and is equal to the attraction of a unit of mass upon a unit of mass placed at a distance

equal to the unit of length. This attraction is exerted in the direction of the line PC. By the principle of the resolution of forces, the force of attraction toward C: the force toward A :: PC : PA :: $\sqrt{a^2+x^2} : a$. Therefore the resolved part of the attraction of an elementary portion of the line AB at C exerted in the direction PA is

$$\frac{knm'adx}{(a^2+x^2)^{\frac{3}{2}}},$$

which is therefore the differential of the attraction of AC for P.

The integral of this expression is

$$\frac{kmm'x}{a\sqrt{a^2+x^2}}+C,$$

which denotes, therefore, the whole attraction of AC upon P exerted in the direction AP.

To determine the value of the constant C, make $x=0$; the attraction of AC will then $=0$. Hence $C=0$.

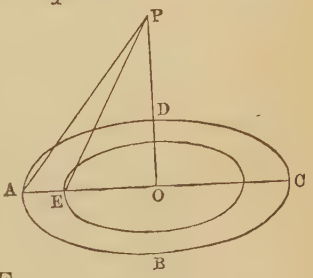
If we make $x=AB$, the above expression becomes

$$\frac{kmm'\times AB}{PA\times PB},$$

which denotes the attraction of the whole line AB for P exerted in the direction AP.

256. *To find the attraction of a uniform circular lamina on a particle situated in a straight line drawn through the centre of the circle perpendicular to its plane.*

Suppose O to be the centre of the circle ABCD, and a particle of matter to be situated at P in the line PO drawn through O perpendicular to the plane of the circle. Take EO, any part of AO, and let another circle be described with O as a centre, and EO as the radius.



Let $PO=a$, and $PE=x$.

Then $EO=\sqrt{x^2-a^2}$.

The area of the circle whose radius is EO is $\pi(x^2-a^2)$, and its differential is $2\pi x dx$. If we put m to denote the mass of a square unit of the lamina, and m' the mass of P, then, as in Art. 255, $\frac{2kmm'\pi x dx}{PE^2}$ will denote the attraction of the particles of the elemental ring for P. But the force of attraction of any particle : the force toward O :: PE : PO. Hence the differential of the attraction of the circle for P exerted in the direction PO is

$$\frac{2kmm'\pi x dx}{x^2}\times\frac{a}{x}=\frac{2kmm'\pi a dx}{x^2}.$$

The integral of this expression is

$$-\frac{2kmm'\pi a}{x}+C.$$

When $x=a$, EO$=0$, and the attraction of the circle whose radius is EO$=0$. Hence

$$0=-2kmm'\pi+C;\text{ that is, } C=2kmm'\pi.$$

Hence the attraction of the circle for P is

$$2kmm'\pi\left(1-\frac{a}{x}\right).$$

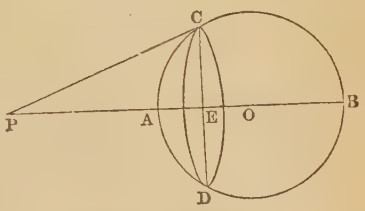
When $x=$PA, the preceding expression becomes

$$2kmm'\pi\left(1-\frac{PO}{PA}\right),$$

which represents the attraction of the whole circle ABCD on the particle P. For greater convenience, we will represent the constant factor $2kmm'\pi$ by KΔ, where Δ represents the thickness of the lamina. Hence the attraction of the circle ABCD on the particle P is $K\Delta\left(1-\frac{PO}{PA}\right)$.

If we suppose the radius of the circle to become infinite, we obtain for the attraction of an infinite lamina on an external particle the constant quantity KΔ; that is, the attraction is independent of the distance of the attracted particle from the lamina, and is directly as the thickness of the lamina.

257. *To find the attraction of a homogeneous sphere upon a particle situated without it.*



Let O be the centre of the sphere, and P the attracted particle. Through P and O draw the line POB, and let CED be a section of the sphere perpendicular to AB.

Let $AO=a$, $PO=b$, $PA=c=b-a$, $PE=y$, $PC=c+x$.

Then $AE=y-c$, and $EB=2a-y+c$.

Now $AE\times EB=EC^2=PC^2-PE^2$;

that is, $(y-c)(2a-y+c)=(c+x)^2-y^2$.

Whence $$y=\frac{2ac+2c^2+2cx+x^2}{2a+2c}$$

$$=\frac{2bc+2cx+x^2}{2b}, \text{ since } b=a+c.$$

Hence, by Art. 256, the attraction of the circle CED upon P is equal to $K\Delta\left(1-\frac{y}{c+x}\right)$,

or $$K\left(1-\frac{2bc+2cx+x^2}{2b(c+x)}\right)dy,$$

or $$K\left(\frac{2ax-x^2}{2b(c+x)}\right)dy.$$

Hence the differential of the attraction of the sphere is equal to

$$\frac{K(2ax-x^2)dy}{2b(c+x)};$$

or, since $dy=\frac{cdx+xdx}{b}$, the differential of the attraction of the sphere is

$$\frac{K(2ax-x^2)dx}{2b^2},$$

whose integral is $$K\times\frac{ax^2-\frac{1}{3}x^3}{2b^2}+C.$$

When $x=0$, the attraction becomes $=0$, and therefore $C=0$. Hence the entire expression for the attraction of the segment ACD upon P is

$$K\times\frac{ax^2-\frac{1}{3}x^3}{2b^2}.$$

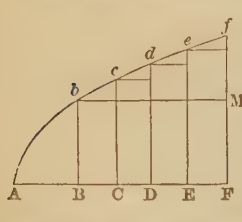
Let $x=2a$, and we shall have the attraction of the whole sphere $=\frac{2a^3\mathrm{K}}{3b^2}$, which varies as $\frac{a^3}{b^2}$. Now the contents of spheres vary as a^3; hence the attraction of a sphere varies as the quantity of matter divided by the square of the distance from the centre. The attraction of a sphere for a particle of matter situated without it is therefore *the same as if all the matter of the sphere were collected into its centre.*

258. *Velocity of a body when the motion is uniform.* When the motion of a body is uniform, the space described in any time is equal to the product of the numbers which express the velocity and the time.

Let v denote the velocity expressed in feet, t the time expressed in seconds, and s the space described. Then $s=tv$; whence $v=\frac{s}{t}$, or the velocity is the quotient of the space divided by the time.

259. *Velocity of a body when the motion is variable.* When the velocity is not constant, it can no longer be measured by the quotient of the space divided by the time; for these quotients are different for different times.

Let the time of motion be represented by the line BF, and let us suppose it to be divided into any number of equal portions BC, CD, DE, etc. Draw the ordinates Bb, Cc, Dd, etc., which shall represent the velocities of the body at the instants corresponding to the points B, C, D, etc.; and complete the parallelograms bC, cD, dE, etc. If the body were to move during the time BC with the velocity which it had at the beginning of that interval, the space described would be proportional to the rectangle bC. Also, if during the time CD the body were to move with the velocity which it had at the beginning of that interval, the epace de-



scribed would be proportional to the rectangle $c\mathrm{D}$; and similarly for the other portions.

Therefore, if the body were to move during each portion of time with the velocity which it had at the beginning of that portion, the whole space described would be proportional to the sum of the rectangles $b\mathrm{C}$, $c\mathrm{D}$, $d\mathrm{E}$, etc.

If the number of these portions of time be increased indefinitely, and consequently the magnitude of each portion be diminished indefinitely, the space thus described will have for its limit the space described in the whole time with the variable velocity. But the sum of the rectangles $b\mathrm{C}$, $c\mathrm{D}$, etc., has for its limit the curvilinear area $b\mathrm{BF}f$; hence the space described in the whole time will be proportional to the curvilinear area $b\mathrm{BF}f$.

Complete the parallelogram $b\mathrm{BFM}$. Then, since

$$b\mathrm{B}\times\mathrm{BF}=b\mathrm{BFM},$$

we have

$$b\mathrm{B}=\frac{b\mathrm{BFM}}{\mathrm{BF}}=\frac{b\mathrm{BF}f}{\mathrm{BF}}\times\frac{b\mathrm{BFM}}{b\mathrm{BF}f}.$$

Now, when BF vanishes, the limit of $\dfrac{b\mathrm{BFM}}{b\mathrm{BF}f}$ is unity. Hence

$$b\mathrm{B}=\text{the limit of }\frac{b\mathrm{BF}f}{\mathrm{BF}}\text{ when BF vanishes.}$$

But $b\mathrm{BF}f$ denotes the increment of the space, and BF the increment of the time. Hence

$$\text{the velocity}=\text{the limit of }\frac{\text{incr. of space}}{\text{incr. of time}};$$

that is,

$$v=\frac{ds}{dt} \qquad (1)$$

260. *To find the accelerating force when it is constant.* When the accelerating force is uniform, the velocity generated in any time is equal to the product of the numbers representing the force and time. Let f be the accelerating force, or the velocity generated in one second. Since the force is uniform, f will be the velocity added in the next second, and $2f$ will be the velocity at the end of two seconds; $3f$ will be the velocity

at the end of three seconds, and tf will be the velocity at the end of t seconds. Hence $v=tf$, or $f=\frac{v}{t}$; that is, the accelerating force is equal to the velocity divided by the time.

261. *To find the accelerating force when it is variable.* When the accelerating force is not constant, it may be shown (as in Art. 259) that it is measured by the limit of the quotient of the increment of the velocity divided by the corresponding increment of the time, when these increments ultimately vanish; that is, the accelerating force = limit of $\frac{\text{incr. of velocity}}{\text{incr. of time}}$,

or
$$f=\frac{dv}{dt}. \qquad (2)$$

If we differentiate Eq. (1), Art. 259, with respect to v, we shall have
$$dv=\frac{d^2s}{dt}.$$

Hence
$$f=\frac{d^2s}{dt^2}. \qquad (3)$$

By combining equations (1) and (2), we find
$$fds=vdv. \qquad (4)$$

Equations (1), (2), (3), (4) are called *the differential equations for force and motion*, and may be applied to the solution of a great variety of problems.

262. *Falling bodies near the earth's surface.* At any given place, and within the range of small elevations, gravity may be considered as a constant accelerating force. Substituting g for f in Eq. (2), we have $dv=gdt$.

Integrating, we have $v=gt+C$.

If the motion of the body commence with the time, then, when $t=0$, $v=0$, and consequently $C=0$. Hence
$$v=gt, \text{ and } t=\frac{v}{g}. \qquad (5)$$

Substituting the value of v in Eq. (1), we have
$$ds=vdt=gtdt.$$

Integrating, we have $$s=\frac{gt^2}{2}+C. \qquad (6)$$

If the motion of the body commence with the time, $C=0$, and we have $$s=\frac{gt^2}{2}.$$

Whence $$t=\sqrt{\frac{2s}{g}}. \qquad (7)$$

From Equations (5), (6), and (7), when either the space, time, or velocity of a falling body is given, the other two quantities can be computed.

263. If we wish to find the space described during any specified portion of the time of fall, it is only necessary to take the integral in Eq. (6) between the required limits. If, for example, we wish to find the space described by a falling body in the 20th second from the beginning of the fall, we have

$$s=\frac{g}{2}(20^2-19^2)=\frac{39g}{2}.$$

If a body be projected vertically downward with a velocity v', the space described in the time t will be

$$s=v't+\frac{gt^2}{2}.$$

If the body be projected vertically upward with a velocity v', the space described in the time t will be

$$s=v't-\frac{gt^2}{2}.$$

264. *Fall of bodies through great distances.* According to Art. 257, the earth's attraction (that is, the force of gravity) varies inversely as the square of the distance from the centre of the earth. Let r = the radius of the earth, g the force of attraction at its surface, a the distance of the body from the centre at the commencement of its motion, and x its distance at the end of t seconds.

Then $$f : g :: \frac{1}{x^2} : \frac{1}{r^2}.$$

Whence $$f = \frac{gr^2}{x^2}. \qquad (8)$$

265. *To find the velocity acquired.*

By Eq. (4), $$f ds = v dv.$$

Whence $$f = \frac{v dv}{ds} = \frac{gr^2}{x^2}, \text{ (from Eq. (8)).}$$

But since $$s = a - x, \; ds = -dx.$$

Hence $$v dv = -\frac{gr^2 dx}{x^2}.$$

Integrating, we have $$v^2 = \frac{2gr^2}{x} + C.$$

The constant C depends upon the velocity which the body has at the distance a where gravity begins to act. If this velocity is zero, then

$$0 = \frac{2gr^2}{a} + C.$$

Whence $$C = -\frac{2gr^2}{a}.$$

Hence the velocity acquired in falling from the initial distance a to the distance x is given by the expression

$$v^2 = \frac{2gr^2}{x} - \frac{2gr^2}{a} = \frac{2gr^2(a-x)}{ax}. \qquad (9)$$

When the body arrives at the surface of the earth (that is, when $x = r$), its velocity will be

$$v = \sqrt{\frac{2gr(a-r)}{a}}.$$

Ex. 1. Find the velocity acquired by a body falling to the earth from the distance of the moon.

Assuming the mean radius of the earth to be 3956 miles, and the distance of the moon equal to 60 radii of the earth, we shall have $a = 60r$. Hence

$$v = \sqrt{\frac{59gr}{30}}.$$

The value of g is $32\frac{1}{6}$ feet. Hence $v = 6.88$ miles per second.

Ex. 2. Find the velocity acquired by a body falling to the earth from the height of 100 miles.

Ans. 5755.9 feet, or 1.09 mile per second.

If a be infinite, then $\frac{a-r}{a}=1$, and we have

$$v=\sqrt{2gr}=6.94 \text{ miles per second.}$$

Therefore the greatest possible velocity acquired by a body in falling to the earth is less than seven miles per second, and a body projected upward with that velocity (supposing there is no resisting medium) would never return.

266. *To find the time of falling.*

By Eq. (1), Art. 259, $\qquad dt=\frac{ds}{v}$.

Substituting $a-x$ for s, and expression (9) for v, we obtain

$$dt=\frac{\sqrt{ax}}{\sqrt{2gr^2(a-x)}}\times d(a-x)$$

$$=-\frac{\sqrt{a}}{r\sqrt{2g}}\times\frac{xdx}{\sqrt{ax-x^2}}.$$

Whence $\qquad t=\frac{\sqrt{a}}{r\sqrt{2g}}\int\frac{-xdx}{\sqrt{ax-x^2}}.$

Integrating by Formula E, Art. 196, we have

$$\int\frac{-xdx}{\sqrt{ax-x^2}}=\sqrt{ax-x^2}-\frac{a}{2}\text{ ver. sin.}^{-1}\frac{2x}{a}+\text{C}.$$

If $t=0$ when $x=a$, we have $\qquad \text{C}=\frac{\pi a}{2}$.

Hence $\qquad t=\frac{\sqrt{a}}{r\sqrt{2g}}\left\{\sqrt{ax-x^2}-\frac{a}{2}\text{ ver. sin.}^{-1}\frac{2x}{a}+\frac{\pi a}{2}\right\}$,

which denotes the number of seconds elapsed in moving from the distance a to the distance x from the centre of the earth. When $x=r$ (that is, when the body arrives at the surface of the earth), the number of seconds elapsed will be

$$t=\frac{\sqrt{a}}{r\sqrt{2g}}\left\{\sqrt{ar-r^2}-\frac{a}{2}\text{ ver. sin.}^{-1}\frac{2r}{a}+\frac{\pi a}{2}\right\}.$$

The last two terms can be united into a single term. Since π − arc whose ver. sin. is A = arc whose cos. is (A − 1),

$$\pi - \text{ver. sin.}^{-1}\frac{2r}{a} = \cos.^{-1}\left(\frac{2r}{a} - 1\right) = \cos.^{-1}\left(\frac{2r-a}{a}\right).$$

Hence $$t = \frac{\sqrt{a}}{r\sqrt{2g}}\left\{\sqrt{ar - r^2} + \frac{a}{2}\cos.^{-1}\left(\frac{2r-a}{a}\right)\right\}.$$

Ex. 1. Find the time required for a body to fall to the earth from the distance of the moon.

Assuming the same values of a, g, and r as in Art. 265, we find $t = 415{,}600$ seconds, which equal 4 days, 19 hours, 26 minutes, and 40 seconds, which is the time a body would be in falling from the moon to the earth's surface.

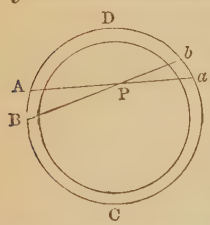
Ex. 2. Find the time required for a body to fall to the earth from a height equal to the radius of the earth.

Ans. 2072 seconds = 34 min. 32 sec.

Ex. 3. Find the time required for a body to fall to the earth from the height of 100 miles. *Ans.* 185 seconds.

If a be infinite, then t is infinite, although the velocity, as found in Art. 265, is less than seven miles per second.

267. *To find the attraction of a homogeneous spherical shell of small thickness on a particle situated within it.* Let ABCD be a spherical shell, and P a particle situated at any point within it. Suppose P to be the vertex of a double cone, whose angle is indefinitely small, and let the cone cut the shell in AB, ab. The inclination of the bases of the cone to the axis of the cone will be the same. Hence the bases of the cones are similar figures; and since their thickness is the same, the quantity of matter in the bases will vary as $AB^2 : ab^2$. Let Q and q represent the quantity of matter in the bases. Then



$$Q : q :: AB^2 : ab^2 :: AP^2 : bP^2;$$

that is,
$$\frac{Q}{AP^2} = \frac{q}{bP^2}.$$

But $\frac{Q}{AP^2}$ and $\frac{q}{bP^2}$ are proportional to the attractions of the bases on the particle P; and since these attractions are equal and in opposite directions, they will not affect the particle P. The same will evidently apply to two spherical polygons *ba* and AB, the bases of any two opposite spherical pyramids, and the whole shell may thus be divided into pairs of elements attracting equally and in opposite directions; and therefore a particle within the shell is equally attracted in every direction, and hence this attraction can have no effect in drawing the particle in any one direction more than in another.

268. *Inside a homogeneous sphere the attraction varies as the distance from the centre.* Suppose a particle situated inside a homogeneous sphere, at the distance x from its centre. Then, by Art. 267, all that portion of the sphere which is at a greater distance than x from the centre produces no effect upon the particle; also, the remainder of the sphere attracts the particle in the same manner as if the mass of the remainder were all collected at the centre of the sphere. Let Q represent the quantity of matter in this sphere. Then the attraction on the particle P is proportional to $\frac{Q}{x^2}$. But Q is proportional to x^3. Hence the attraction on the particle varies as $\frac{x^3}{x^2}$, or x; that is, the attraction varies as the distance from the centre of the sphere.

269. *To determine the velocity acquired by a body falling beneath the surface of the earth.* Suppose an opening to be made from the surface to the centre of the earth, and that a body moves along it under the influence of the earth's attraction. By Art. 268, the accelerating force f varies directly as the distance from the centre. Let r denote the radius of the earth, g the force of gravity at the surface, and s any variable distance from the surface. Then $r-s$ will be the distance from the centre, and we shall have

$$f:g::r-s:r,$$

or $$f=\frac{g(r-s)}{r}.$$

But by Eq. (3), Art. 261,

$$f=\frac{d^2s}{dt^2}=\frac{g(r-s)}{r}.$$

Multiplying by $2ds$, we obtain

$$\frac{2dsd^2s}{dt^2}=\frac{2g}{r}(r-s)ds.$$

Integrating, we obtain

$$\frac{ds^2}{dt^2}=v^2=-\frac{g}{r}(r-s)^2+\mathrm{C}.$$

To determine the constant C, suppose the body to start from the surface without any initial velocity. Then, when $s=0$, $v=0$.

Hence $$0=-gr+\mathrm{C},$$

or $$\mathrm{C}=gr.$$

Therefore $$v^2=\frac{g}{r}\{r^2-(r-s)^2\}, \qquad (1)$$

or $$v=\sqrt{\frac{g}{r}\{r^2-(r-s)^2\}},$$

from which we may deduce the value of v for any point in the line of the body's motion.

If we make $s=r$, this equation becomes

$$v=\sqrt{gr},$$

which is the velocity acquired by the body when it has arrived at the centre of the earth. From this point the body, in consequence of its acquired velocity, will ascend to the surface, and, having moved through an entire diameter, we shall have $s=2r$, which, substituted in the above equation, gives $v=0$; that is, the velocity will be 0. It will therefore again return through the centre, and so on, making an infinite number of oscillations, each of equal duration.

Ex. 1. Compute the velocity acquired by a body in falling from the surface to the centre of the earth.

Ans. 4.91 miles per second.

Ex. 2. Compute the velocity acquired by a body in falling from the surface of the earth to the depth of 100 miles.

Ans. 1.0968 mile per second.

270. *To determine the time of a falling body beneath the surface of the earth.*

From Eq. (1), Art. 269, we have

$$\frac{ds^2}{dt^2}=\frac{g}{r}\{r^2-(r-s)^2\}.$$

Hence
$$\sqrt{\frac{g}{r}}.dt=\frac{ds}{\sqrt{r^2-(r-s)^2}}.$$

Integrating, we obtain

$$t=\sqrt{\frac{r}{g}}\times\int\frac{ds}{\sqrt{r^2-(r-s)^2}}$$

$$=\sqrt{\frac{r}{g}}\times\cos.^{-1}\frac{r-s}{r}+C.$$

When $s=0$, $t=0$, and hence $C=0$.

When $s=r$, we have $t=\frac{\pi}{2}\sqrt{\frac{r}{g}}$,

which denotes the time of falling from the surface of the earth to the centre. The time of falling through the diameter will be

$$t=\pi\sqrt{\frac{r}{g}}.$$

Ex. 1. Compute the time required for a body to fall from the surface to the centre of the earth. *Ans.* 21 min. 6 sec.

Ex. 2. Compute the time required for a body to fall from the surface of the earth to the depth of 100 miles.

Ans. 181.6 seconds.

THE END.